Longitudinal Control Algorithm for Automated Vehicle Merging

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Abstract

This paper considers longitudinal control of automated vehicle merging in a mathematical approach for Automated Highway Systems. Merging maneuver is defined as one vehicle in the merging lane to be inserted in the middle between two vehicles in the main lane at fixed merging point which is the intersection of these two lanes. The main lane vehicles can change speed. To achieve this, the merging vehicle must properly adjust its speed and acceleration such that it reaches the merging point at right time with the same speed and acceleration as the main lane vehicles. This problem is a little similar to but different from the well-known missile interception problem. The longitudinal control problem is proposed for different road layouts, based on which a unified mathematical model is established. Then a new concept, virtual platooning, is introduced, which effectively avoids a two-point boundary value problem. Based on this concept, an analytic solution with mathematical proof is provided. It is also discretized as a recursive algorithm for real-time use. A dynamic real-time simulation is published at PATH Website. This algorithm has been successfully implemented with automated cars.

1 Introduction

Nowadays, heavy traffic problem is getting more and more prominent in highway driving. Automated highway systems (AHS) will be a promising alternative to improve traffic throughput and safety (Shladover, 1995; Varaiya, 1993). Efficient and reliable merging maneuver in traffic management and control for AHS will definitely increase traffic throughput and reduce traffic congestion and accident.

This paper presents a newly developed real-time algorithm with theoretical analysis for automated vehicle merging in AHS from a control point of view. By definition, automated merging maneuver is properly and safely to insert a vehicle from on-ramp in the middle between two pre-selected vehicles of a platoon in main lane. The crux of the algorithm is to generate a reference trajectory for the merging vehicle based on the position and speed of the vehicles of a platoon in the main lane. Once a reference trajectory for the merging vehicle is available, a proper controller can be used to make the merging vehicle to track it.
Related research works in longitudinal transition control (Connolly and Hedrick, 1999; Narendran et al., 1992) considered joining of a vehicle to the end of a platoon. This maneuver is for a single vehicle to join a platoon of vehicles in the same lane from behind. Although they referred to that maneuver as "merging", it is in fact a different maneuver from the merging developed here. (Rajamani et al., 2000) considered the implementation of several automated vehicle maneuvers including splitting, joining and lane changing, but not merging. It is emphasized that none of previous work has addressed the merging problem that we consider in this paper.

For safety, a deterministic approach is adopted in the merging algorithm. From this viewpoint, the problem of one vehicle on the on-ramp merging with a platoon of vehicles in the main lane can always be abstracted as the entrance of the merging vehicle between two pre-selected vehicles in the platoon at a fixed merging point which is the intersection of those two lanes. Only the platoon of these two vehicles are directly relevant to the merging vehicle. The leader vehicle in the platoon has a speed $v_p(t)$ and acceleration $a_p(t)$, which also represent platoon speed and platoon acceleration respectively. Several points are emphasized.

(1) The merge manual requires a combination of decisions and control actions at both the regulation layer and the coordination layer of the five-layer control hierarchy that Varaiya defined for AHS (Varaiya, 1993).

(2) The choice of the two relevant vehicles is determined by a roadside manager (coordination layer). After making the decision, the roadside manager passes the rest of the merging task to the relevant vehicles (the two prefixed vehicles in the main lane and the merging vehicle in the merging lane).

(3) A separate algorithm is used for a splitting maneuver of the two relevant vehicles and those following them in the platoon such that there is a proper gap for the merging vehicle to enter when the merging vehicle reach the merging point. The splitting maneuver trajectory planning can be found in (Connolly and Hedrick, 1999). The constant spacing platoon forming and platoon keeping can be found in (Hedrick, 1998; Lu et al., 2001; Swaroop and Hedrick, 1999).

(4) Most importantly, this algorithm is actually a trajectory planning for the merging vehicle to adjust its speed to ensure it to reach the merging point at right time, speed and acceleration to form a new platoon of $n + 1$ vehicles. To provide a reference speed is better than to provide a reference acceleration in that the former is suitable for both speed and distance control which are crucial for vehicle platooning.

Let $v(t)$ and $a(t)$ denote the speed and acceleration of the merging vehicle (the vehicle on the merging lane) respectively. For safety and passenger's comfort, it is required that at the time instant $T_{merg}$ of merging, the following two conditions should be satisfied

(i) \begin{align}
    v(t_{merg}) &= v_p(t_{merg}) \\
    a(t_{merg}) &= a_p(t_{merg})
\end{align} \hspace{1cm} (1.1)

(ii) The relative distances between the merging vehicle and the two vehicles of the platoon in main lane are the same as the desired following distance $l_{des-follow}$, which implies that the merging vehicle must be inserted in the middle.

Obviously, three factors should be taken into consideration in control: acceleration, speed and distance. There are two major difficulties in this trajectory planning. One is the differences in road
layouts. The other is the main lane vehicle speed variation. In general, only two typical road layouts which can practically appear: (a) there is no parallel section between the merging lane and the main lane (only one crossed point); (b) there is a parallel section between them.

Obviously, the control task for case (a) is more demanding because the merging is required to be fulfilled in a very short time period. To cope with different road layouts, a new concept, i.e. virtual platooning, is introduced, which essentially shifts the time instant for platoon forming forward before real merging starts. This gives the merging vehicle more flexibility to adjust its speed and acceleration to \( v_p(t) \) and \( a_p(t) \) as well as the relative distances to \( l_{de:s\_follow} \). Mathematically, this effectively avoids a two-point-boundary-value problem. As is known that the solution (only numerical solution is available) to such a problem cannot be implemented in real-time. This is one highlight of the algorithm.

To overcome the difficulty caused by \( v_p(t) \) changing, an adaptive merging algorithm is proposed. Essentially, the reference speed of the merging vehicle changes according to the speed of the leader vehicle (the first of the two prefixed vehicles) in the main lane. Meanwhile it takes into account the distance requirement for merging . Its acceleration demanding for the longitudinal controller can be adjusted by proper choice of parameters and is thus more practical and robust in practice. This is the second highlight of the algorithm.

The third highlight is that the generated reference speed \( v_{md}(t) \) has the same smoothness property as that of \( v_p(t) \) -- the leader vehicle speed in the main lane. This is important for real-time implementation and safety.

The merging problem is a little similar but different from the well-known missile interception problem. Similarity: merging vehicle behaves as a missile and the target is the middle point between the two pre-selected main lane vehicles; Differences: The motion trajectory of the missile in the space is generally unknown but it is unnecessary to require the boundary conditions (1.1) to hold. For merging, the space trajectory is known in advance, which is determined by the lane, but the constraints (1.1) are required for safe platoon forming, which is not necessary for missile interception.

This algorithm has been successfully implemented and tested at different maximum speeds using automated cars (Lu et al., 2000). A dynamic real-time simulation for vehicle speed \( \leq 90[\text{mph}] \) is available at PATH Website (Lu, 2000).

This paper is organized as follows. After introduction, control problem for merging is presented in section 2, mathematical modeling for merging is presented in Section 3, and an algorithm (solution) with discretization for real-time use is presented and proved in section 4. Finally, some concluding remarks are presented in Section 5.

## 2 Control Problem Formulation

Pertinent factors for controller design of vehicle merging in the regulation layer are mainly the following three:

- (a) Geometric layout of the road;
- (b) Signal measurement on each vehicle and communication between relevant vehicles;
- (c) Traffic situation and vehicle speed in the main lane.

Vehicles in the main lane are supposed to be grouped in series of platoons. A roadside manager is
responsible for the coordination of these platoons (Lu, 2002). Under this assumption, the two relevant vehicles in the main lane can be determined in advance by the roadside manager. This consideration makes it possible to isolate the merging problem.

2.1 Nomenclature

Variables and parameters used in the longitudinal control algorithm for merging are listed below:

\[ P_i, \ i = 1, 3 \] — vehicle ID on main lane
\[ P_2 \] — vehicle ID on merging lane
\[ v(t) \] — merging vehicle speed, measurable
\[ v_{md}(t) \] — desired speed of merging vehicle, to be determined
\[ v_p(t) \] — speed of the leader vehicle in the platoon on main lane, measurable
\[ x(t), x_{md}(t) \] — relative distance and reference relative distance between two vehicles, which means the virtual distance between the leader vehicle and the merging vehicle here

\[ t_{30} \] — time instant for \( P_1 \) passing the virtual starting point on main lane
\[ t_{30} \] — time instant for \( P_2 \) passing the virtual starting point on merging lane
\[ t_{merge} = \max(t_{30}, t_{30}) \] — time instant for longitudinal merging control algorithm to start
\[ T_{merge} \] — time instant when merging is completed (real merge)
\[ T_{virt} \] — time instant when a virtual platoon is formed but merging is not completed
\[ Q_{start, 1}, Q_{start, 2} \] — vehicle positions on main lane and merging lane when \( t = t_{merge} \), respectively
\[ Q_{virt, 1}, Q_{virt, 2} \] — virtual positions in main lane and merging lane at which virtual platoon is formed before practical vehicle merging happens

\[ l_0 \] — desired distance between the two consecutive vehicles in the main lane (for the merging vehicle to enter)

\[ l_{des, follow} \] — desired distance between consecutive vehicles in platoon after merging
\[ l_i \] — physical length of vehicle \( P_i, \ i = 1, 2, 3 \).
\[ Q_1 \] — a point in main lane marked by magnets using special coding (detectable on vehicle)
\[ Q_2 \] — a point in merging lane marked by magnets using special coding (detectable on vehicle)
\[ Q_0 \] — crossing (merging) point of main lane and merging lane
\[ |Q_1 Q_0| \] — the distance between \( Q_1 \) and \( Q_0 \).

It is noted that merging maneuver starts means that the merging algorithm starts to activate, while merging starts means the real merging, i.e. the merging vehicle arrives at the merging point and enters in the middle between the two prefixed vehicles.

2.2 Geometric Layouts of the Road

Two different geometric road layouts which represent all possible practical cases lead to slightly different problem formulation. The main lane is the one nearest to the on-ramp.

In some highway, there is no parallel lane for the merging vehicle to speed up before merging. For convenience, it is called layout A as in Figure 1. In this case, \( Q_1, Q_2, Q_0 \) are marked by specially coded magnets. At the time instant when \( P_1 \) or \( P_2 \) pass them, the magnetometer will acknowledge the vehicles. This information can be used to determine the time instant and the vehicle position with respect to the merging point \( Q_0 \). Once both \( P_1 \) and \( P_2 \) have passed \( Q_1 \) and \( Q_2 \) respectively, merging maneuver starts. Here the difficulty is that the merging point \( Q_0 \) is fixed. The merging vehicle has to
arrive at this point at a right time instant and at proper speed and acceleration for a merging to be fulfilled safely and successfully.

In freeway, there is usually a short lane to help the merging vehicle to speed up. This can be abstracted as a parallel section between the main lane and the merging lane, which is called layout B as in Figure 2.

Similarly, $Q_1$ and $Q_{10}$ are marked by specially coded magnets on main lane and $Q_2$, $Q_{10}$ and $Q_{20}$ can be marked on merging lane. This road layout has more flexibility because merging can actually be carried out at any point between $Q_{10}$ and $Q_{20}$ in a much longer time period compared to the previous road layout. It is thus more flexible to adjust the merging time instant and the speed and acceleration of the merging vehicle.

Although the merging problems for these two different road layouts seem different, from a control design viewpoint, an algorithm suitable for road layout A will automatically apply to layout B.

2.3 Merging and Platooning

From a control design viewpoint, in an AHS, all the merging maneuvers can be described as merging of a single vehicle with a platoon of vehicles of size $n$ which is known in advance. It is assumed that there is a platoon speed $v_p$ which is the real-time speed for the leader vehicle in the platoon. Vehicle following strategy for the platooning is that each other vehicle except the leader will follow the vehicle in its immediate front. Thus the relative distance of the two relevant vehicles in the platoon can be kept constant as required around the time instant when practical merging happens. Now the general case can be divided into three according to the relative position of the merging vehicle with respect to the platoon at the moment of merging:

1. Merging in front of a platoon of size $n$: No vehicle is closely in front of the platoon. In this case, one can put a virtual leader vehicle in front of the real leader vehicle. It can be simplified as one vehicle in the merging lane enters between the virtual vehicle and the real leader vehicle.

2. Merging behind a platoon of vehicles of size $n$: No vehicle is closely behind the platoon. This case can be abstracted as merging with only one vehicle in the main lane from behind. Similarly, one can put a virtual vehicle behind the platoon. Thus the problem can be simplified as the merging vehicle enters between the last vehicle in the platoon and the virtual vehicle.

3. Merging in between two vehicles in a platoon of size $n$: The relative position of the merging vehicle in the platoon after merging is fixed before the merging maneuver starts. This can be arranged by a roadside manager and acknowledge the relevant vehicles by communication. Magnetic marker on both main lane and merging lane will help to make the decision.

Eventually, all the three cases are reduced to the situation that one vehicle in the merging lane is to enter in between two vehicles in a platoon on main lane. After merging, $n + 1$ vehicle platoon is formed. The algorithm presented here is for this general case.

Merging vehicle entering a specified position in a platoon of vehicles in the main lane is an important generalization beyond the merging maneuver proposed in (Varaiya, 1993) where the merging maneuver is defined as a merging vehicle merges to the end of a platoon. This provides the advantages of increased design and operating flexibility for the system. Even for the case of merging behind a platoon, since the arrival time, speed and acceleration at the merging point in between two platoons need to be controlled, the algorithm developed here is also indispensable. In fact, a platoon can be abstracted
as a ”single vehicle” and inserting a merging vehicle in between two platoons is equivalent to inserting a merging vehicle in between the two abstracted ”single vehicles”.

3 Mathematical Modeling for Merging

The main problem for vehicle merging is to determine a desired synthetic acceleration for the merging vehicle from which throttle control command and brake control command can be determined. The desired synthetic acceleration is eventually determined by the desired speed \( v_{md}(t) \) for a controller. This \( v_{md}(t) \) should be determined by the speed \( v_p(t) \) of the leader vehicle in the platoon in the main lane as well as relative positions of the relevant vehicles at the time instant when merging maneuver starts. The following mathematical model gives the conditions (equations and initial conditions) that \( v_{md} \) should satisfy.

The key step here is to determine the merging time instant \( T_{merg} \) from which \( v_{md}(t) \) is determined. For convenience, one can assume that

\[
|Q_{\text{start}_1} Q_0| - |Q_{\text{start}_2} Q_0| + l_1 + l_{\text{des}_\text{follow}} > 0
\]

which can always be satisfied by proper arrangement of specially coded magnets at \( Q_{\text{start}_1} \) and \( Q_{\text{start}_2} \). The control problem is formulated as follows.

**Longitudinal Control Problem for Merging:** Design a reference trajectory for merging vehicle \( P_2 \) such that

\[
\begin{align*}
   v_{md}(t_{\text{merg}}) &= v(t_{\text{merg}}) \\
   v_{md}(T_{\text{merg}}) &= v_p(T_{\text{merg}}) \\
   a_{md}(T_{\text{merg}}) &= a_p(T_{\text{merg}})
\end{align*}
\]

(1) three vehicle \( P_1, P_2, P_3 \) form a platoon of speed \( v_p(t) \) for \( t \geq T_{\text{merg}} \)

(2) the relative distance between two consecutive vehicles is the same as the prescribed distance \( l_{\text{des}_\text{follow}} \)

The following conditions are assumed:

(a) the platoon of vehicles \( P_1 \) and \( P_3 \) with fixed distance is formed and maintained by a separate controller;

(b) \( v_p \) is measured and passed to \( P_2 \) and \( P_3 \) by communication in real-time;

(c) the distance \( |Q_{1} Q_0|, |Q_{2} Q_0| \) and \( |Q_{2} Q_{01}| \) and \( |Q_{2} Q_{02}| \) are known in advance;

(d) the time instant \( t_{01} \) when \( P_1 \) passes \( Q_1 \) can be measured and all are available for \( P_2 \) by communication. The time instant \( t_{02} \) when \( P_2 \) passes \( Q_2 \) is measured by \( P_2 \) itself. Thus merging maneuver starting time instant \( t_{\text{merg}} \) and position \( Q_{\text{start}_1}, Q_{\text{start}_2} \) can be calculated on \( P_2 \).

3.1 Road Layout A

As in Figure 1, there is no parallel section between the main lane and the merging lane. \( T_{\text{merg}} \) is determined by the following rule
\[
\int_{t_{\text{merg}}}^{T_{\text{merg}}} v_p(t) dt = |Q_{\text{start}} - Q_0| + l_1 + l_{\text{des\textunderscore follow}} = L_1
\]

\[v_{\text{md}}\] is then determined by

\[
\int_{t_{\text{merg}}}^{T_{\text{merg}}} v_{\text{md}}(t) dt = |Q_{\text{start}} - Q_0|
\]

The modeling problem is first discussed in a special case when \(v_p(t)\) is constant and then the general case.

### 3.1.1 Main Lane Vehicle Speed Fixed

In this case, the platoon speed is constant, i.e. \(v_p(t) = \text{const}\) and \(T_{\text{merg}}\) is thus known before merging maneuver starts.

\[
T_{\text{merg}} - t_{\text{merg}} = L_1/v_p
\]

The trajectory planning problem becomes: To find a reference trajectory \(v_{\text{md}}(t)\) such that

\[
\begin{align*}
v_{\text{md}}(t_{\text{merg}}) &= v(t_{\text{merg}}) \\
\int_{t_{\text{merg}}}^{t_{\text{merg}} + L_1/v_p} v_{\text{md}}(t) dt &= |Q_{\text{start}} - Q_0| \\
v_{\text{md}}(t_{\text{merg}} + L_1/v_p) &= v_p \\
a_{\text{md}}(t_{\text{merg}} + L_1/v_p) &= 0
\end{align*}
\]

(3.3)

The first equation are initial condition. The second is the merging requirement for distance compatibility. The last two are the platoon forming requirement. It is clear that if one of the following conditions

\[
v_p(t_0) = v(t_0) \\
L_1 = v_p(t_0) (T_{\text{merg}} - t_{\text{merg}})
\]

is violated, there is no trivial solution \(v_{\text{md}}(t) = \text{const}\).

### 3.1.2 Main Lane Vehicle Speed Changing

If \(v_p(t)\) is time varying, \(T_{\text{merg}}\) is unknown. The problem is to find a reference trajectory \(v_{\text{md}}(t)\) such that

\[
\begin{align*}
v_{\text{md}}(t_{\text{merg}}) &= v(t_{\text{merg}}) \\
\int_{t_{\text{merg}}}^{T_{\text{merg}}} v_p(t) dt &= |Q_{\text{start}} - Q_0| + l_1 + l_{\text{des\textunderscore follow}} = L_1 \\
\int_{t_{\text{merg}}}^{T_{\text{merg}}} v_{\text{md}}(t) dt &= |Q_{\text{start}} - Q_0| \\
v_{\text{md}}(T_{\text{merg}}) &= v_p(T_{\text{merg}}) \\
a_{\text{md}}(T_{\text{merg}}) &= a_p(T_{\text{merg}})
\end{align*}
\]

(3.4)

Again, the first equation is an initial condition. The second and the third conditions represent distance compatibility. From the second equation, one can determine \(T_{\text{merg}}\) implicitly. The third equation determines \(v_{\text{md}}(t)\) once \(T_{\text{merg}}\) is known. The last two equations are the platoon forming requirement. Clearly, two implicitly coupled two boundary value functional equation need to be solved to get a solution. It is also expected that the solution here is not unique.
3.2 Road Layout B

As in Figure 2, one way to formulate the problem is this. Use the same model developed above for road layout B except that the merging point $Q_0$ can be any point between $Q_{01}$ and $Q_{02}$. Thus (3.4) can still be used for this road layout. It is noted that, the flexibility in choosing $Q_0$ greatly reduces the difficulty in solving the problem. This point will become clear in next section.

3.3 Unified Modeling for Two Road Layouts

For road layout A, there is no reason that a platoon should be formed only after merging has completed. The idea is that, before $P_1$ and $P_2$ arrive at the merging point, a virtual platoon can be formed.

A virtual platoon means that at some time instant $T_{virt}$

$$t_{merg} \leq T_{virt} \leq T_{merg}$$

$P_1$ arrives at $Q_{virt,1}$ as shown in Figure 3. At this point, the following conditions are satisfied:

$$v_{md}(T_{virt}) = v_p(T_{virt})$$
$$a_{md}(T_{virt}) = a_p(T_{virt})$$

$$|Q_{virt,1}Q_0| + l_1 + l_{des\_follow} = |Q_{virt,2}Q_0|$$

Then for $t \in [T_{virt}, T_{merg}]$, it is sufficient that

$$v_{md}(t) = v_p(t)$$
$$a_{md}(t) = a_p(t)$$

for a real platoon to be formed at the time instant when $P_2$ arrives at $Q_0$. It is noted that, as for road layout A, the concept of virtual platooning brings more flexibility in finding a solution in theoretical analysis. It also greatly increases safety by avoiding the last second jump. In this way, the longitudinal control problem of merging for these two seemingly quite different road layouts can be unified. The condition corresponding to (3.4) can be stated as follows:

There exist time instant $T_{virt}$ and point $Q_{virt,1}$ between $Q_1$ and $Q_0$ ($Q_{01}$ for road layout B) and $Q_{virt,2}$ between $Q_2$ and $Q_0$ ($Q_{02}$ for road layout B) such that

$$v_{md}(t_{merg}) = v(t_{merg})$$
$$\int_{t_{merg}}^{T_{virt}} v_p(t)dt = |Q_{start,1}Q_{virt}|$$
$$\int_{t_{merg}}^{T_{virt}} v_{md}(t)dt = |Q_{start,2}Q_{virt}|$$

$$|Q_{virt,1}Q_0| + (l_1 + l_{des\_follow}) = |Q_{virt,2}Q_0|$$

$$v_{md}(T_{virt}) = v_p(T_{virt})$$
$$a_{md}(T_{virt}) = a_p(T_{virt})$$

Remark 1 Any trajectory planning for automated vehicle merging should satisfy conditions (3.6) to avoid last minute jump and thus to improve safety. This is a special type of two point-boundary-value problem (Keller, 1968).
Next section will present some adaptive solutions of the problem, which are particularly suitable for real-time implementation.

4 Adaptive Solutions

This is a speed based real-time algorithm. It uses a variable structure approach to design a reference speed \( v_{md}(t) \) for the merging vehicle, which has the same smoothness property as that of \( v_p(t) \). The smoothness of \( v_{md}(t) \) is important in practical implementation because non-smooth of \( v_{md}(t) \) will cause jumps in the calculation of \( \frac{d}{dt}(v_{md}(t)) \) when fed into the longitudinal controller, which might cause unease of passengers and real-time instability.

First a special solution is presented for (3.3) when \( v_p(t) \) is constant. Then a general solution for (3.6) is presented and proved. The special case is presented here to help the readers to understand the difficulty in the general case.

4.1 A Special Case: \( v_p(t) \) is Constant

The following reference speed \( v_{md}(t) \) for the merging vehicle can be adopted if \( v_p(t) \) is nearly constant. This is special case of Theorem 2. Thus a proof to Theorem 2 is a natural proof for Theorem 1.

**Theorem 1** Suppose that

1. Time response of a known longitudinal controller is fast enough and speed and distance tracking error is small enough, i.e.

\[
\begin{align*}
    a(t) & \approx a_{md}(t) \\
    v(t) & \approx v_{md}(t) \\
    x(t) & \approx x_{md}(t)
\end{align*}
\]

for \( t \in [t_{merg}, T_{merg}] \);

2. \( |Q_1Q_0| \) and \( |Q_2Q_0| \) are long enough, or equivalently, the time interval \([t_{merg}, T_{merg}]\) is long enough for the merging vehicle to adjust its speed and distance;

3. The distances between maneuver start points to the merging point satisfy the following condition

\[
    dist_{para} = |Q_{start,1}Q_0| - |Q_{start,2}Q_0| + l_1 + l_{des\_follow} > 0;
\]

4. The platoon speed \( v_p(t) \) is constant and there exists \( \delta > 0 \) such that

\[
    v_p(t) > v(t_{merg}) + \delta, \quad t \in [t_{merg} + \eta, T_{merg}]
\]

holds, where \( 0 < \eta << 1 \);

5. The following reference speed for \( P_2 \) is fed into the longitudinal controller

\[
    v_{md}(t) = \begin{cases} 
    (1 - \sin \left( \frac{\pi(t-t_{merg})}{2\delta} \right)) v(t_{merg}) + \sin \left( \frac{\pi(t-t_{merg})}{2\delta} \right) v_p(t), & t_{merg} \leq t < t_\delta \\
    (1 - \alpha) v_p(t_\delta) + \alpha v_p(t), & t_\delta \leq t < T_virt \\
    v_p(t), & T_virt \leq t \leq T_{merg} \\
    \alpha \left( \frac{|Q_{start,2}Q_0|}{|Q_{start,2}Q_0| + l_{des\_follow}} \right)^\beta, & \beta > 0
    \end{cases}
\]

(4.4)
Then virtual platoon is guaranteed to be formed for some $\beta$, i.e. there exist $\beta > 0$, a time instant $T_{\text{v1}} \in [t_{\text{merged}}, T_{\text{merged}}]$ and points $Q_{\text{v1}}$ and $Q_{\text{v2}}$ such that

$$|Q_{\text{virt,1}}Q_0| \leq |Q_1Q_0|, \quad i = 1, 2$$

and all the conditions in (3.4) are satisfied.

**Remark 2.** First of all, the reference speed $v_{\text{md}}(t)$ is divided into 3 phases according to time periods. Physical meaning of each phase is explained as follows.

Phase 1. For $t_{\text{merged}} \leq t < t_{\delta}$, it is to bring the speed $v(t)$ close to platoon speed $v_p(t)$ such that the joining point to the second phase at time instant $t = t_{\delta}$ is smooth. This phase can be characterized as both speed and distance control. If $|v(t_{\text{merged}}) - v_p(t_{\text{merged}})|$ are small, this phase is naturally ignored.

Phase 2. For $t_{\delta} \leq t < T_{\text{v1}}$, it is the essential part of this trajectory planning. The purpose is to adjust both speed $v(t)$ and distance $|Q_{\text{v1}}Q_0|$ such that the formation of the virtual platoon has been completed at the end of this phase. It is essentially speed control only.

Phase 3. For $T_{\text{v1}} \leq t \leq T_{\text{merged}}$, it is a virtual platoon keeping control. As long as the merging vehicle follows the speed and acceleration of the platoon in main lane, real merging will be guaranteed at the time instant $T_{\text{merged}}$.

$\alpha$ is chosen time invariant but its parameter $\beta$ needs to be determined, which will be considered later.

### 4.2 General Case: $v_p(t)$ is time varying

In real traffic, $v_p(t)$ can be time varying to some extent. The following theorem gives an adaptive algorithm for this general case.

**Theorem 2** Suppose that

(1) time response of the known longitudinal controller is fast enough and speed and distance tracking error is small enough, i.e.

$$a(t) \approx a_{\text{md}}(t)$$

$$v(t) \approx v_{\text{md}}(t)$$

$$x(t) \approx x_{\text{md}}(t)$$

for $t \in [t_{\text{merged}}, T_{\text{merged}}]$;

(2) $|Q_1Q_0|$ and $|Q_2Q_0|$ are long enough, or equivalently, the time interval $[t_{\text{merged}}, T_{\text{merged}}]$ is long enough to adjust the speed and distance of the merging vehicle;

(3) The positions of the specially coded magnets and vehicle length satisfy the following condition

$$\text{dist}_{\text{para}} = |Q_{\text{start,1}}Q_0| - |Q_{\text{start,2}}Q_0| + l_1 + l_{\text{des, follow}} > 0;$$

(4) For $t \geq t_{\text{merged}}$, there holds

$$v_p(t) - v(t_{\text{merged}}) \geq 0$$

and there exists a $T \in (t_{\text{merged}}, T_{\text{merged}}]$ such that

$$\int_{t_{\text{merged}}}^{t} \left( v_p(s) - v(t_{\text{merged}}) \right) ds \geq 2 \text{dist}_{\text{para}}$$
(5) The following reference speed for $P_2$ is fed into the longitudinal controller

$$v_{md}(t) = \begin{cases} 
(1 - \alpha(t)) v(t_{\text{merg}}) + \alpha(t) v_p(t), & t_{\text{merg}} \leq t \leq T_{\text{virt}} \\
v_p(t), & T_{\text{virt}} < t \leq T_{\text{merg}}
\end{cases}$$

$$\alpha_0(t) = \frac{\int_{t_{\text{merg}}}^{t} v_p(s) ds}{\int_{t_{\text{merg}}}^{t} v(s) ds + \text{dist}_{\text{para}}}$$

$$\alpha(t) = \alpha_0(t), \quad \beta > 0$$

(4.8)

Then virtual platoon is guaranteed to be formed for some $\beta > 0$, i.e. there exist a constant $\beta > 0$, a time instant $T_{\text{virt}} \in [t_{\text{merg}}, T_{\text{merg}}]$ and points $Q_{\text{virt,1}}$ and $Q_{\text{virt,2}}$ such that

$$|Q_{\text{virt,1}} Q_0| \leq |Q_i Q_0|, \quad i = 1, 2$$

and all the conditions in (3.6) are satisfied. Besides, $v_{md}(t)$ has the same smoothness property as $v_p(t)$.

**Remark 3** Before proving the theorem, physical meaning of the algorithm and the conditions in the theorem are briefly explained as follows.

(a) Obviously, no phase 1 of previous algorithm appears here because $\alpha(t_{\text{merg}}) = 0$ is not necessarily true. A smooth connection needs to be formed here adaptively. Other two phases are similar except that $\alpha(t)$ is time variant, which is closely related to the speed of $P_1$ and $P_2$. The reference trajectory can be divided into two phases according to time:

Phase 1. $t_{\text{merg}} \leq t \leq T_{\text{virt}}$, this is the essential part of this trajectory planning. The purpose is to adjust both speed $v(t)$ and distance $|Q_{\text{virt,1}} Q_0|$ such that the formation of the virtual platoon actually completed at the end of this phase.

Phase 2. $T_{\text{virt}} < t \leq T_{\text{merg}}$, a virtual platoon keeping control. As long as the merging vehicle follows the speed of the platoon in main lane, real merging will be guaranteed at time instant $T_{\text{merg}}$.

(b) The assumptions in the theorem are basically the physical constraints which can be set as required in practice. Condition (1) depends on the controller adopted. Conditions (2) and (3) can be implemented by proper infrastructure of magnet marking in both merging lane and main lane. Condition 3 is mainly to avoid singularity. Condition (4) is a reasonably weak condition. It requires that, main lane platoon vehicle speed during merging is not lower than the speed of the merging vehicle at the time instant $t_{\text{merg}}$, which is reasonable for general traffic. In fact, dist_{para} can be chosen as 0.2, for example. If not so, one can reduce the speed of merging vehicle somehow before merging maneuver starts to make it satisfied.

**Proof of Theorem 2.** Clearly $\alpha(t_{\text{merg}}) = 0$ and so $v_{md}(t_{\text{merg}}) = v(t_{\text{merg}})$. Thus the first condition in (3.6) is satisfied.

It is sufficient to prove that $T_{\text{virt}}$ and $Q_{\text{virt}}$ exist such that the rest of the conditions in (3.6) are satisfied. Equivalently, it is necessary and sufficient to prove that there exists a time instant $T_{\text{virt}} \in [t_{\text{merg}}, T_{\text{merg}}]$ such that

$$\alpha(T_{\text{virt}}) = 1$$

or equivalently

$$\alpha_0(T_{\text{virt}}) = 1$$
In fact, \( \alpha_0(T_{\text{virt}}) = 1 \) if and only if
\[
\int_{t_{\text{merg}}}^{T_{\text{virt}}} v_p(s) ds = \int_{t_{\text{merg}}}^{T_{\text{virt}}} v(s) ds + \text{dist}_\text{para}
\] (4.9)
Or equivalently
\[
|Q_{\text{start,1}}Q_{\text{vert,1}}| = |Q_{\text{start,2}}Q_{\text{vert,2}}| - (l_1 + l_{\text{ke.start,flow}})
\] (4.10)
i.e.
\[
|Q_{\text{vert,1}}Q_0| + l_1 + l_{\text{ke.start,flow}} = |Q_{\text{vert,2}}Q_0|
\] (4.11)
This is exactly the distance compatibility condition, the 4th in (3.6).

The 5th condition in (3.6) is clearly satisfied. To prove the 6th condition in (3.6), i.e. the continuous of acceleration, is satisfied, differentiae \( v_{\text{md}}(t) \) to obtain
\[
\dot{v}_{\text{md}}(t) = -\alpha(t) v(t_{\text{merg}}) + \dot{v}(t) v_p(t) + \alpha(t) a_p(t)
\] (4.12)
Because
\[
\alpha(T_{\text{virt}}) = 1.
\] (4.13)
It is sufficient to prove that
\[
[-\alpha(t) v(t_{\text{merg}}) + \dot{v}(t) v_p(t)]_{t=T_{\text{virt}}} = 0
\] (4.14)
or equivalently
\[
\dot{\alpha}(t) \bigg|_{t=T_{\text{virt}}} = 0
\]
because
\[
v_p(t) - v(t_{\text{merg}}) \geq \delta
\]
by assumption. Now
\[
\dot{\alpha}(t) = \alpha_0^{\beta-1}(t) \dot{\alpha}_0(t)
\]
and
\[
\dot{\alpha}_0(t) \bigg|_{t=T_{\text{virt}}} = \frac{v_p(t) \left( \int_{t_{\text{merg}}}^{l_{\text{merg}}} v(s) ds + \text{dist}_\text{para} \right) - v(t) \int_{t_{\text{merg}}}^{l_{\text{merg}}} v_p(s) ds}{\left( \int_{t_{\text{merg}}}^{l_{\text{merg}}} v(s) ds + \text{dist}_\text{para} \right)^2} \bigg|_{t=T_{\text{virt}}}
\] (4.15)
\[
\approx \frac{v_p(t) \left[ \int_{t_{\text{merg}}}^{l_{\text{merg}}} v(s) ds + \text{dist}_\text{para} - \int_{t_{\text{merg}}}^{l_{\text{merg}}} v_p(s) ds \right]}{\left( \int_{t_{\text{merg}}}^{l_{\text{merg}}} v(s) ds + \text{dist}_\text{para} \right)^2} \bigg|_{t=T_{\text{virt}}} = 0
\]
because
\[
v(T_{\text{virt}}) \approx v_{\text{md}}(T_{\text{virt}}) = v_p(T_{\text{virt}})
\]
\[
\alpha_0(T_{\text{virt}}) = 1.
\]
Now it is to prove the existence of \( T_{\text{virt}} \) such that \( \alpha(T_{\text{virt}}) = 1 \) by contradiction. Suppose that such a \( T_{\text{virt}} \) does not exist in \([t_{\text{merg}}, t_{\text{merg}}]\).
First, it is noted that the following facts are true:
(a) \( \alpha_0(t_{\text{merg}}) = 0 \).
(b) \( \alpha_0(t) \) is continuous for \( t \in [t_{\text{merg}}, T_{\text{merg}}] \) because \( \int_{t_{\text{merg}}}^{t} v_p(s)ds \) and \( \int_{t_{\text{merg}}}^{t} v(s)ds \) are continuous.

Condition (3) implies that \( 0 \leq \alpha_0(t) \). Thus if there exists \( t_1 \in [t_{\text{merg}}, T_{\text{merg}}] \) such that \( \alpha_0(t_1) > 1 \), then there exists \( T_{\text{virt}} \in [t_{\text{merg}}, T_{\text{merg}}] \) such that \( \alpha_0(T_{\text{virt}}) = 1 \) because \( \alpha_0(t) \) is continuous. By assumption, it must be true that \( \alpha_0(t) < 1 \), \( t \in [t_{\text{merg}}, T_{\text{merg}}] \). Then there exists a \( \mu > 0 \) such that \( \alpha_0(t) \leq 1 - \mu \) because \( \alpha_0(t) \) is continuous and \( [t_{\text{merg}}, T_{\text{merg}}] \) is a closed interval. Thus for \( 0 < \varepsilon < \frac{1}{3} \), there exists \( \beta \) such that

\[
\alpha(t) = \alpha_0^\beta(t) \leq \varepsilon
\]

Compare the numerator and the denominator of \( \alpha_0(t) \) as follows.

\[
\int_{t_{\text{merg}}}^{t} v(s)ds \approx \int_{t_{\text{merg}}}^{t} v_{\text{md}}(s)ds \\
= \int_{t_{\text{merg}}}^{t} [v(t_{\text{merg}}) + \alpha(s)(v_p(s) - v(t_{\text{merg}}))] ds
\]

On the other hand, (4) implies that

\[
\int_{t_{\text{merg}}}^{t} \alpha(s)(v(s) - v(t_{\text{merg}})) ds \\
\geq \int_{t_{\text{merg}}}^{t} \varepsilon (v(s) - v(t_{\text{merg}})) ds > 0
\]

The denominator has the estimation

\[
0 \leq \int_{t_{\text{merg}}}^{t} v(s)ds + \text{dist}_{\text{para}} \\
\leq \int_{t_{\text{merg}}}^{t} (v(t_{\text{merg}}) + \varepsilon v_p(s) - \varepsilon v(t_{\text{merg}})) ds + \text{dist}_{\text{para}}
\]

The difference between the denominator and the numerator of \( \alpha_0(t) \) is

\[
\int_{t_{\text{merg}}}^{t} v(s)ds + \text{dist}_{\text{para}} - \int_{t_{\text{merg}}}^{t} v_p(s)ds \\
\leq \int_{t_{\text{merg}}}^{t} ((1 - \varepsilon)v(t_{\text{merg}}) + \varepsilon v_p(s)) ds + \text{dist}_{\text{para}} - \int_{t_{\text{merg}}}^{t} v_p(s)ds \\
= \int_{t_{\text{merg}}}^{t} (\varepsilon - 1) (v_p(s) - v(t_{\text{merg}})) ds + \text{dist}_{\text{para}} \\
\leq 2(\varepsilon - 1) \text{dist}_{\text{para}} + \text{dist}_{\text{para}} \\
= -\frac{1}{3} \text{dist}_{\text{para}} < 0
\]

This is a contradiction. i.e. if \( T_{\text{virt}} \) is large enough, \( \alpha_0(T_{\text{virt}}) < 1 \) is violated. This contradiction implies the existence of \( T_{\text{virt}} \) such that \( \alpha_0(T_{\text{virt}}) = 1 \).

Furthermore, if \( v_p(t) \) is continuously differentiable \( n \) times, then \( \alpha(t) \) is continuously differentiable \( n + 1 \) times. Thus \( v_{\text{md}}(t) \) and \( v_p(t) \) have the same smoothness property.

This completes the proof. \( \diamond \)
The parameter $\beta$ will affect the acceleration of merging vehicle. Its choice is not unique. Thus the adaptive solution above is not unique. One may ask the question: Is it possible to find an optimal reference trajectory for merging vehicle with respect to some performance index? The answer is not obvious. The difficulty here is that optimization will lead to a nontrivial two point boundary value problem which certainly brings a great difficulty in finding a solution or in the implementation even if a solution can be proved to exist. However, one can expect to choose $\beta$ properly to reduce the demanding for the acceleration of the merging vehicle. This will be discussed in future work.

### 4.3 Discretization

For real-time application, it is necessary to discretize the algorithm. Suppose that time step used for real-time control is $\Delta t > 0$.

$$
t_i = t_{\text{merg}} + i \Delta t
$$

$$
i = 0, 1, \ldots
$$

(4.8) can be discretized as a recursive algorithm as follows.

$$
v_{md}(t_i) = v_{md}(t_{\text{merg}}) = v(t_{\text{merg}})
$$

$$
(1 - \alpha(t_i)) v(t_{\text{merg}}) + \alpha(t_i) v_p(t_{i-1}), \quad t_{\text{merg}} \leq (t_{\text{merg}} + i \Delta t) \leq T_{\text{virt}}
$$

$$
v_p(t_{i-1}), \quad T_{\text{virt}} < (t_{merg} + i \Delta t) \leq T_{\text{merg}}
$$

$$
\alpha(t_i) = \frac{\alpha_0(t_i)}{\sum_{j=1}^{i} v_p(t_{j-1}) \Delta t}
$$

$$
\alpha_0(t_i) = \frac{\sum_{j=1}^{i} v_p(t_{j-1}) \Delta t}{\sum_{j=1}^{i} v_p(t_{j-1}) \Delta t + \text{dist}_{\text{para}}}
$$

It is this recursive algorithm that has been implemented (Lu et al., 2000; Lu, 2000).

### 5 Concluding Remarks

This paper considers a longitudinal control problem for automated vehicle merging. Mathematically, it is a little similar but different from the well-known missile interception problem. First, a longitudinal control problem is proposed based on performance requirements. Second, a mathematical model is established, which characterizes the merging problem. Any merging algorithm for automated vehicles’ safe and smooth merging should satisfy the model. Then an adaptive algorithms is given as the solution of the problem, which is suitable for real-time control and for different road layouts. To unify merging algorithm for different road layouts, a concept of virtual platooning is proposed. Essentially, a virtual platoon in the following sense is to be formed before merging vehicle arrives at the merging point:

1. Speed, acceleration of the merging vehicle in the merging lane are the same as those of the platoon vehicles in the main lane;

2. Relative to the merging point, the distance of the merging vehicle in the merging lane is the same as the middle point between $P_1$ and $P_3$ in the platoon in the main lane.

This algorithm generates a reference trajectory for the merging vehicle based on the speed of the leading vehicle in the main lane. It has the same smoothness property as that of the platoon speed.
This algorithm has been successfully implemented and tested using automated cars. Practical test also showed that the choice of $\beta$ is very important to system stability and performance of the merging vehicle.

References


Figure 1: Road layout A
Figure 2: Road layout B
Figure 3: A unified road layout