Trip Assignment – a literature review

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1. GENERAL BACKGROUND

In transportation system analysis, a transportation system is often simplified into a form of network and zoning systems. The term *network* refers to a structure in which there are two types of elements: a set of nodes and a set of links that join some pairs of nodes. In a detailed network model, the nodes in a network model represent individual road junctions. Each link corresponds to a section or road; in more aggregate nodes, links can represent collections of roads. The topology of network is specified by the presence or absence of links between nodes which determine the possibility of travel from one place to the other. There are several attributes associated with each link in the network model to define the characteristics of that link. The most commonly used attributes include link length, free flow travel time, and link capacity. With these attributes, the delay, number of stops and travel time on each link can then be estimated according to the flow of traffic carried by the link. Various functional forms have been proposed to model the relationship between link travel times and traffic flows. In general, because an increase in link traffic flow will normally decrease the travel speed along the link, travel times are usually considered to be positive monotonic increasing functions of traffic flow. Parameters in the travel time functions often include free flow travel times (i.e. link travel times when there is no traffic on links) and link capacities (i.e. maximum values of traffic flow along the link). Some examples of travel time functions can be found in Patriksson (1994, p20). In addition to links, the term *route* or *path* is defined to represent a sequence of directed links leading from one node to another. The corresponding travel time along a route can be determined as the sum of the travel times along the links comprising that route, within which each of the link travel time is calculated according to their corresponding time of entry.

The term *zone* in the zoning system refers to a partition of an urban area. Within each of these zones, various data can be collected for calibrating and validating the transport model. These data include demographic features of people in the zone and levels of economic activity including employment, shopping space, educational and recreational facilities (Ortúzar and Willumsen, 2001). Each zone is represented in the network by a special node called a *centroid*. Each centroid can either be an *origin* node from which traffic enters the network, or a *destination* node to which traffic leaves the network.

After building a representation of the transport system, analysis and planning procedure can then be carried out. The classic procedure of analysis and planning in transport practice, known as the *four-stage model*, is shown in Figure 1. The four stages are trip generation, trip distribution, modal split, and assignment. The four-stage model starts with estimating the total number of trips generated by each zone based on the data of the levels of economic activity in that zone. The next stage is to distribute these trips from their origins to particular destinations. The following stage, modal split is an estimation of the choice of transport modes, such as car, underground train, or bus, of the trips. The final stage, assignment, is to estimate how the trips travel through the network, the traffic flows generated, the resulting traffic conditions, and the costs of travel for each origin-destination pair. A detailed discussion on the four-stage model can be found in Ortúzar and Willumsen (2001).

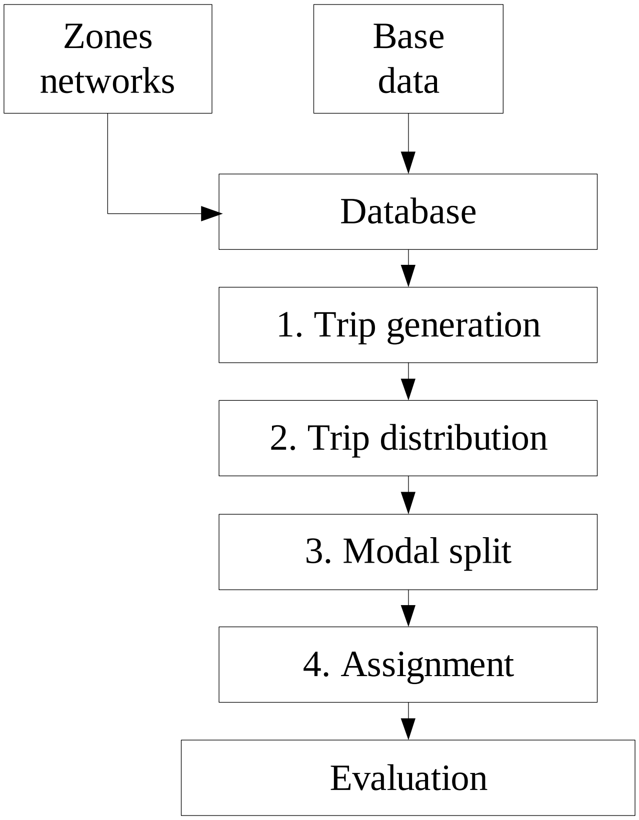

Figure 1 The classic four-stage transport planning model

2. TRAFFIC ASSIGNMENT

A traffic assignment model aims to estimate how traffic flows through a road system and the associated effects of traffic on the system. These effects can be measured by a number of criteria including distance travelled, travel time, delay, fuel consumption and environmental pollution. Traffic assignment models can also be used to investigate the responses of traffic to changes in the system (for example, changes in travel demand, travellers' information, road capacities, signal timings, and road tolls).

Formulating and solving a traffic assignment model requires three kinds of information. The two of these are the demand for travel and the characteristics of transport system. The demand for travel, which is estimated by the three earlier stages of the four-stage model, represents the likely travel decisions that travellers would make, given the performance of the transport system. Following the first three steps in the four-stage model, the travel decisions considered include choices of destination, mode, frequency of trip, and even whether to travel at all (IHT, 1997, p91). It should be noted that although population, land-uses, and other factors could vary over time, so does the travel demand. Conventional planning models only consider the travel demand within a particular period of time and the demand is regarded as time-independent throughout that time period. The second component of a traffic assignment formulation is a network model of the characteristics of transport system. The function of this network model is to define the relationship between the travel demand and the performance of the transport system. For example, travel times are modelled as increasing with travel demand, due to the decreases in travel speeds of vehicles (IHT, 1997, p91).

Given the demand for travel and the characteristics of a transport system, the third kind of information is a way of estimating the corresponding distribution of the travel demand over the transport system. The most widely accepted way is through the two principles of traffic assignment proposed by Wardrop (1952).

2.1 Wardrop's first principle – user equilibrium

Wardrop adopted the supply-demand equilibrium concept of economics, which suggests that travel demand should be balanced against the performance of the transport system in servicing that level of demand. This gives Wardrop's (1952) first principle, or the *user equilibrium* principle:

> "the journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route."

The underlying assumption of this principle is that all travellers are supposed to choose their routes of travel through the network according to the common criterion that their individual journey times are minimized. In addition, all travellers will experience the same journey times if they encounter identical traffic conditions. Furthermore, all travellers will have perfect information on all possible routes through network, no matter whether the routes are used or not.

In fact, this concept of equilibrium is found to be a powerful tool for analysing transport system, as Bell and Iida (1997) wrote:

> "While a transport system may never actually be in a state of equilibrium, it is assumed that it is at least near equilibrium, tending towards equilibrium, and only prevented from attaining equilibrium by changes in external factors… At equilibrium, the transport system reduces to a fixed point (the equilibrium costs and flows), and powerful analytical techniques … exist for finding the fixed point. Proponents of equilibrium theory take it as a matter of faith that, given the existence of an equilibrium, there are behavioural mechanisms that push the transport system to this fixed point."

A traffic assignment model should be formulated in mathematical terms before it can be analysed and solved numerically. User equilibrium traffic assignment can be stated equivalently as the following complementary inequality for the route flow e_p:

$$e_p \begin{cases} > 0 \Rightarrow C_p = C_{od}^* \\ = 0 \Rightarrow C_p \geq C_{od}^* \end{cases} \qquad \forall p \in P_{od}, \quad \forall od \tag{1}$$

where e_p is the flow assigned to route *p*, P_{od} is the set of all routes from origin *o* to destination *d*, C_p is the travel time along route *p*, and C_{od}^* is the minimum travel time from *o* to *d*.

Beckmann et al. (1956) were the first to transform the user equilibrium principle into a mathematical programming problem for link flow v_a.

$$\min_{\mathbf{v}} \sum_{a \in L} \int_{v=0}^{v_a} c_a(v)dv \tag{2}$$

subject to

$$\sum_{p \in P_{od}} e_p = E_{od} \qquad \forall od \tag{3}$$

$$e_p \geq 0 \qquad \forall p \in P_{od} \quad \forall od\,. \tag{4}$$

The notation v_a represents the flow of traffic on link *a*, $c_a(\cdot)$ is the travel time along link *a* and is a function of link flow v_a, E_{od} represents the traffic flow between origin *o* to destination *d*. It is noted that the objective function is formulated in terms of link flows, while the constraints are formulated in terms of route flows. Hence, the following definitional constraint is required to inter-relate the link flows and route flows

$$v_a = \sum_{od} \sum_{p \in P_{od}} e_p \delta_p^a \qquad \forall a\,, \tag{5}$$

where δ_p^a is a indicator variable:

$$\delta_p^a = \begin{cases} 1 & \text{if link } a \text{ is on route } p \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

Constraint (5) is also part of the optimization program (2).

Beckmann (1956) showed that solving this mathematical programming formulation is equivalent to solving the static user equilibrium assignment problem (1). The equivalency can be proven by verifying that the Karush-Kuhn-Tucker (KKT) necessary conditions for a minimum point of the problem (see Sheffi, 1985, pp63 – 66; Patriksson, 1994, pp35-36) are exactly the conditions of user equilibrium. Since its introduction, the transformation technique by Beckmann (1956) is now standard and well-known in transport literature, and hence its mathematical details are not shown here for brevity. A range of efficient solution algorithms were later developed, and they can be employed to solve Beckmann et al.'s (1956) mathematical programming formulation and its extensions effectively. Examples of the algorithms can be found in Evans (1976), Lee (1995), and Bar-Gera (2002).

Uniqueness of equilibrium solution

The uniqueness of user equilibrium solution depends on two conditions: the feasible region and the objective function in Beckmann's formulation have both to be convex (Sheffi, 1985, p66). Given the linearity of the constraints, the resultant feasible region of the optimization problem will be convex. If the travel cost function c_a is monotonic increasing with respect to link flow v_a, then it will imply the integral and hence the objective function (1-2) is strictly convex. As a result, the equilibrium solution with respect to *link flow* is unique. Nevertheless, one should note that the above analysis is only confined to link flow, and indeed the equilibrium solution is not unique with respect to *path flow*. An illustrative example can be referred to Sheffi (1985, p68)[*].

2.2. Wardrop's second principle – system optimum

Although user equilibrium may be a good representation of distribution of existing network traffic, such distribution of traffic generally does not lead to the best possible use of the network system. This is because user equilibrium considers that each individual traveller is acting only in their own interests, but not necessarily in the interest of the system as a whole.

In fact, the discrepancy between the behaviour of individual travellers acting on their own interests and the interests of the whole community is known in economic literature as the "divergence between private cost and social cost". It was first raised and discussed by Pigou (1920) and Knight (1924). In accordance with this observation, Wardrop (1952) further proposed his second principle of traffic assignment to describe how travellers could be allocated centrally to minimize the total cost incurred by all travellers. Wardrop's (1952) second principle or the system optimal principle is:

> "the average journey time is a minimum."

Under system optimum, some travellers may be assigned to routes that have costs higher than the minimal that they could travel along. This is because the additional costs incurred by such travellers will be outweighed by the greater savings that accrue to the others. The user equilibrium and system optimal principles will produce identical results when the network is uncongested (Sheffi, 1985, p72). Although the system optimal assignment is not a realistic representation of network traffic, it provides a bound on how we can make the best use of the road system, and as such it is a useful benchmark for evaluating various traffic control policies. Using economic terminology, the user equilibrium and the system optimal assignments respectively represent the descriptive (positive) and normative representations of traffic flow patterns on road networks.

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[*] Such non-uniqueness with respect to path flow also applies for dynamic traffic assignment with either point-queue or physical-queue paradigms (Szeto and Lo, 2004).

The system optimal assignment can also be formulated mathematically as a static minimization problem of the total system journey time spent in the network:

$$\min_{\mathbf{v}} \sum_{a \in L} v_a c_a(v_a) \tag{7}$$

subject to constraints (1-2) – (1-4).

The optimality conditions of the system optimal assignment are given in Sheffi (1985, pp69 – 72) as

$$e_p \left\{ \begin{aligned} > 0 \Rightarrow C_p + \sum_{\forall a} \delta_p^a v_a \frac{\partial c_a(v_a)}{\partial v_a} = C_{od}^* \\ = 0 \Rightarrow C_p + \sum_{\forall a} \delta_p^a v_a \frac{\partial c_a(v_a)}{\partial v_a} \geq C_{od}^* \end{aligned} \right\} \qquad \forall p \in P_{od}, \quad \forall od . \tag{8}$$

The quantity $C_p + \sum_{\forall a} \delta_p^a v_a \frac{\partial c_a(v_a)}{\partial v_a}$ is interpreted as the marginal contribution of an additional traveller on route p to the total travel time on that route p. The derivative of link travel time with respect to the link flow , $\frac{\partial c_a(v_a)}{\partial v_a}$, represents the additional travel time induced by an additional traveller to each of the existing travellers on the link. When the transport network is at system optimum, this *marginal* travel time on all used routes connecting each origin-destination pair in the network is equal. In other words, to optimally operate a transportation system in an equilibrium state, each traveler on each link is expected to pay for a toll equalling $\frac{\partial c_a(v_a)}{\partial v_a}$. This is the underlying concept of *marginal pricing* in the literature of transport engineering and economics.

2.3. Traffic assignments and network games

There have been comparisons in the literature between the traffic assignments and the game theory in economics.

In game theory, a game is considered to be a potential game if the incentive of all players to change their strategy can be expressed in one global function, the potential function (Monderer and Shapley, 1994). The fact that Wardrop equilibrium can be obtained by using an equivalent optimization problem makes it classified as a kind of potential games (Altman and Wynter, 2004).

The fact that in a Wardrop game where there is only a finitely number of players (travelers) also makes it to be classified called congestion games (Patriksson, 1994; Altman and Wynter, 2004). Congestion games were first defined by Rosenthal (1973). Rosenthal (1973) considered that each player in the congestion game to be an individual traveler. The travelers can choose pure strategies (i.e. make their route choices to their respective destination) to minimize their payoff function (i.e. individual travel time). Such congestion game was shown to be equivalent to a non-cooperative, pure strategy Nash game. In other words, Wardrop equilibrium can be regarded as a kind of Nash equilibrium.

2.4. Day-to-day evolution towards equilibrium

The original Wardrop's analysis disregards how an equilibrium state is actually reached over time. This produces a number of studies on the "day-to-day" travelers' behavior adjustments and the evolution of the overall system towards equilibrium. Some pioneers in the transportation literature includes Horowitz (1984), Cantarella and Cascetta (1995), and Watling (1999).

In the literature of computer science, Fischer et al. (2006) studied the rerouting policies in a dynamic round-based variant of Wardrop game. Fischer et al. (2006) imagined that the travelers play a repeated game in rounds. In each round, each traveler may compare the latency of his current route with the latency of another route and switch to the other route if it promises a better latency. The problem with this natural approach is that other travelers might switch to the same route simultaneously so that latency of a traveler may not improve or even get worse. Following this, the game may get struck in oscillations. Fischer et al. (2006) considered how a large population of travelers compute and learn an equilibrium more efficiently based on simple sampling and adaptation policies. Their studies can contribute to the design of distributed adaptive re-routing protocols, such as deployment of various intelligent transportation system (ITS) technologies, which quickly converge to stable routing allocations. Fischer et al. (2006) showed that a simple replication protocol in which travelers adopt strategies of more successful travelers reaches equilibrium within a time bound that is independent of the size and the structure of the network but only dependent on the travel costs of travelers. They further showed that to achieve convergence to Wardrop equilibrium, one also needs an exploration component discovering possibly unused strategies in addition to replication. Fischer et al. (2006) investigated a sampling based replication-exploration protocol and its convergence time to equilibrium.

Along the same line, Friesz et al. (2004) and Yang et al (2007) looked into the evolution of system optimal assignment. In particular, Yang et al (2007) pointed out that with the traditional marginal cost pricing (see Equation 8), the overall system performance could show deterioration over some period during the transition before reaching its ultimate system optimal state. Such interim deterioration is not desired by transportation planners. Yang et. al (2007) demonstrated how to deploy a day-to-day dynamic toll such that the system performance can improve continuously over "days". Yang et al. (2007) further proposed a steepest-descent day-to-day toll such that the transportation system can converge to system optimal at the fastest rate.

3. WITHIN-DAY DYNAMIC TRAFFIC ASSIGNMENT MODELS

On the characteristics of the transport system, the steady-state assignment model considers traffic flows and travel times to be time-invariant and travel demand to be below the physical capacity of the transport network. However, traffic flows and travel times are dynamic in nature. In addition, there is a possibility that during some parts of the day, the travel demand will exceed the capacity of the network. This temporary overloading cannot be represented by static models in a satisfactory manner (Heydecker and Addison, 2005).

On the travel demand side, the steady-state traffic assignment model specifies the demand for travel to a particular time period under consideration, and treats it as constant over that period of the day. This treatment could mask any systematic variation in travel demand over time of the day. Indeed, empirical studies (Hendrickson and Plank, 1984; Small, 1992) confirmed that travellers do change their times of departure subject to the traffic conditions they encounter, especially during morning and evening peak periods. This temporal variation in travel demand, which is known as the *peak spreading* phenomenon, cannot be captured by steady-state traffic assignment models.

Vickrey (1969) was among the first proposing dynamic model of transportation system which is called *bottleneck model*. In the bottleneck model, traffic congestion was assumed to take the form of queuing behind a bottleneck of fixed flow capacity, on a single travel link connecting a single origin-destination pair. In the model, each identical traveller was assumed to commute in his or her own car from home (i.e. origin) to work (i.e. destination) along the single travel link. All travellers wish to arrive at work at the same time, which is impractical to achieve because the capacity of the bottleneck is finite. As a result, some travellers have to arrive earlier and some arrive later. The cost of arriving earlier or later than the desired arrival time was called *schedule delay cost.* Each traveller will make his or her choice of time of departure in order to minimize the associated *total travel cost,* which is essentially a cost associated with the time spent on travel plus the schedule delay cost. The equilibrium is achieved if all travel can be made at the same total travel cost. This means that in equilibrium, travellers will trade off changes in schedule delay costs against those in travel time. Those who travel off-peak so as to achieve short journeys do so at the expense of travelling at relatively unfavourable times, which is represented through schedule delay costs. On the other hand, those who arrive close to their desired time do so at the expense of a relatively long journey. Following Vickrey (1969), authors including Yagar (1971), Hurdle (1974), and Merchant and Nemhauser (1978a; b) have acknowledged the importance of Vickrey's (1969) work and have contributed to the dynamic transport models.

Nevertheless, the significance of the within-day dynamic models was not widely acknowledged until the inception of intelligent transportation systems and the technological advances in traffic control systems in the early 1990s. The application of intelligent transport systems (ITS) has shown their ability to improve transport networks in many ways by providing information and guidance to travellers. The benefits of ITS include reducing travel times in lightly-loaded conditions, and increasing capacity and hence reducing travel times in more heavily loaded ones. For example, Adler (2001) showed how travel times could be reduced by about 1 minute in a 15-minute journey through providing advanced traffic information and route guidance. Rajamani and Shladover (2001) showed that ITS technologies could be used to provide autonomous adaptive cruise control systems that increase road capacity from about 2,000 to about 3,000 vehicles per lane-hour. A more detailed review on ITS can be referred to Heydecker (2002a). In addition to ITS, designing and implementing innovative traffic control systems and policy also require dynamic traffic assignment models to estimate travellers' likely response. Some examples of these control strategies include network access control (see for example, Smith and Ghali, 1990a; b; Lovell and Daganzo, 2000; Erera et al., 2002), network design and road capacity management (see for example, Ghali and Smith, 1993; Arnott, De Palma and Lindsey, 1993, 1998; Heydecker, 2002b), and time-varying road pricing (see for example, Yang and Huang, 1997; Wie and Tobin, 1998; Ettema et al., 2006). Due to these genuine needs, dynamic traffic assignment problems have become a popular and important research topic in both academia and industry in the last two decades.

Following Wardrop's (1952) principles, within-day dynamic traffic assignment can be formulated through two approaches: dynamic user equilibrium and dynamic system optimal assignments.

3.1. Within-day dynamic user equilibrium

In the literature, dynamic user equilibrium assignment has been being the focus of research. The formulations of dynamic user equilibrium assignment can be grouped into five categories:

1. Mathematical programming (see for example, Janson, 1991; Ran and Boyce, 1996; Han and Heydecker, 2006);
2. Non-linear complementarity problem (see for example, Wie et al., 2002);
3. Fixed point problem (see for example, Addison and Heydecker, 1993; Heydecker and Addison, 1996);
4. Variational inequality (see for example, Friesz et al., 1993; Ran and Boyce, 1996; Szeto and Lo, 2004).

Following the success in tackling static traffic assignment problem, much work on dynamic user equilibrium assignment attempted to use a mathematical programming approach. Janson (1991) proposed a mathematical programming formulation by integrating Beckmann's (1956) equilibrium objective function with respect to time. However, as later pointed out by Lin and Lo (2000), and Boyce, Lee and Ran (2001), the formulation of Janson's (1991) mathematical programme cannot capture the traffic dynamics, the temporally asymmetric nature of dynamic traffic cost functions, and the time-dependent interaction between traffic flows and travel times. Lin and Lo (2000) also showed with simple counter-example that solving Janson's (1991) formulation does not necessarily lead to a solution that satisfies a dynamic user equilibrium condition. Recently, Han and Heydecker (2006) have reformulated Beckmann's (1956) mathematical programme and have addressed the problem raised by Lin and Lo (2000) in Janson's (1991) formulation. However, Han and Heydecker's (2006) formulation can be too cumbersome for practical implementation. In addition, their formulation has yet to be applied to networks in which interactions between flows from different origin-destination pairs are involved.

Friesz et al. (1993) were the first to formulate and analyse dynamic user equilibrium traffic assignment problem using variational inequalities. As shown by Patriksson (1994) and Nagurney (1993), variational inequalities can be regard as a generalization of mathematical programming, non-linear complementarity problem, and fixed point problem. Due to their generality, variational inequalities have attracted a lot of attention as a means of formulating and analysing dynamic traffic assignment. Detailed discussions on formulation of variational inequality can be found in Friesz et al. (1996), Ran and Boyce (1996), and Nagurney (1993).

Cell-based dynamic user equilibrium assignment

Lo and Szeto formulated a CTM based dynamic user equilibrium assignment with route choice (Lo and Szeto, 2002) and a dynamic user equilibrium assignment with combined route choice and departure time choice (Szeto and Lo, 2004) by using the variational inequality (VI) with an underlying cell transmission model. The formulation was solved by using a projection method developed by Han and Lo (2003). Han and Lo's (2003) algorithm will be convergent if the solution set of the VI problem is nonempty. Numerical results were given in both papers where physical features (such as spilling back) of traffic were captured and good equilibration results of traffic assignment were shown.

One should note that with the physical-queue paradigm, an important recent finding is that the existence of equilibrium assignment solutions is no longer guaranteed (Daganzo, 1998; Szeto and Lo, 2005). Examples have been found in Szeto and Lo (2005) which show that under congested conditions, the route travel time functions may become discontinuous, making it impossible in certain cases to find an user equilibrium solution.

3.2. Within-day dynamic system optimum

Dynamic user equilibrium is used to represent the distribution of traffic that arises when travellers consider their own interests alone. However, as discussed previously, such distribution of traffic generally does not lead to the best possible use of the transport system, because the user equilibrium considers that each individual traveller is acting only in their own interests, rather than those of the community. Dynamic system optimal assignment, in contrast, considers that there is a central *system manager* distributing the traffic over time within a fixed horizon so that the total, rather than individual, benefit of all travellers in the system is maximised.

Analytical dynamic system optimal assignment is an important yet underdeveloped area and indeed it is one of the most challenging areas in transportation research. Different from its static counterpart, dynamic system optimal assignment is a dynamic optimization problem, which aims to calculate an *optimal time path* for the decision variables instead of a single *optimal value* as in the static case. As noted by Dorfman (1969), such a problem is difficult to solve and "is not for beginners". In addition, the challenges associated with dynamic system optimal assignment problem are also due to the range of interrelated requirements on their components (i.e. travel demand, characteristics of road system, and the way in which traffic is distributed) to perform in a satisfactory manner. As twelve years ago, Patriksson (1994) wrote:

> "So far, no well-founded dynamic models free from any serious anomaly such as instant propagation of some travellers, infinite cycling, failure to recognize the first-in-first-out

> principle, etc., have appeared, and their numerical solution most often rely on a time-discretization which brings the dynamic model into a (typically very large) static one."

Merchant and Nemhauser (1978a; b) were the first to formulate and analyse dynamic system optimal assignment. Merchant and Nemhauser's (1978a; b) formulation was then followed and modified by many others (see for example, Ho, 1980; Carey, 1987; Friesz et al., 1989; Yang and Huang, 1997; Wie and Tobin, 1998). However, these previous studies used an outflow traffic model, whose plausibility was later found to be questionable for its violation of causality. Addison and Heydecker (1998) used an alternative calculus of variations technique to analyse and calculate the system optimal assignment with departure time choice. However, the calculus of variations is complicated to use and to implement.

Friesz et al. (2006) and myself (Chow, 2007) formulated the dynamic system optimal assignment with departure time choice as a *state-dependent* optimal control problem. Our objective was to minimize the total system travel cost by controlling the inflow to the network over time. To ensure the plausibility[†], the objective function was subject to a set of constraints: the proper flow propagation along each link, flow conservation, amount of total throughput between each origin-destination pair, non-negativity of traffic. One technical difficulty arose due to the duration of the time lag between changes to the control variable (route inflows) and the corresponding response (route outflows) depends on the state variables (traffic volume on the routes). This state-dependent control theoretic formulation is unorthodox in the control theory literature. The corresponding optimality conditions were derived by using a calculus of variations technique. Essentially we both came up with a similar optimality condition which can be regarded as a dynamic extension of traditional system optimality condition (see Sheffi, 1985). We showed that the dynamic system optimal assignment can be reduced to an equivalent dynamic user equilibrium assignment formulation in which several additional costs are introduced to each route and departure time in use. The additional cost can be regarded as the dynamic externality which is the sensitivity of the value of the value of objective function (i.e. total system travel cost) with respect to a changes in the route inflow profile. One has to pay attention that, different from the analysis in static case, such externality is *dynamic* in nature which means a change in the route inflow at a particular time will has an effect on traffic at that time *and times thereafter*.

Cell-based dynamic system optimum assignment

Ziliaskopoulos (2000) formulated a CTM based dynamic system optimal assignment with route choice for many-to-one networks by using the linear programming (LP) approach with an underlying cell transmission model. Ziliaskopoulos's formulation aimed to minimize the total system travel time, which was shown to be equivalent to minimize the cell occupancy at each time interval, within a fixed time planning horizon. The control (decision) variables were the inflow to each route between each origin-destination pair. The state (response) variables were the cell occupancies during each time interval. The associated dual variables would represent the changes in the value of the objective function (i.e. total system travel time) with respect to changes in occupancy in the corresponding cell within the corresponding time interval. In economic terminology, the dual variables can be regarded as the marginal cost with respect to

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[†] You may refer the appendix for some discussion on the plausibility of traffic model.

traffic in each cell. Necessary and sufficient conditions of the optimization problem were derived and proven. Numerical results were given in both paper where physical features (such as spilling back) of traffic were captured. Nevertheless, Ziliaskopoulos (2000) assumed that traffic could be held *anywhere* at *anytime* for the benefit of the whole system. Such *holding back* problem was criticized to be unrealistic and is needed to be remedied[‡].

4. SOME POSSIBLE EXTENSIONS: TRAVEL UNCERTAINTIES AND HETEROGENEITY

On the travellers' behaviour, the above analysis supposes that all travellers have perfect information on the traffic conditions that they will encounter on their journeys. However, it is understood that travellers do not have such information in reality. Investigating the effects of *uncertainties* is an important future extension. One popular way to capture uncertainties in travel is by adding stochastic terms in the travel cost functions to represent the uncertainties in travel information obtained by travellers (see for example, Sheffi, 1985; Lim and Heydecker, 2005; Maher, et al., 2005). In addition to the realism, such stochastic traffic assignment models have also been shown in the literature to have certain computational advantage over the deterministic ones. Several studies (see for example, Ying and Yang, 2005; Connors et al., 2007) have shown that incorporating the stochastic terms has a desirable consequence of providing smoothness and convexity to both demand and travel performance functions for analysis and solution algorithms to work with.

Furthermore, the work above considers travelers have same value of travel time and time-specific costs, while it is also not exactly the case in reality. Taking the *heterogeneity* among travelers into account is necessary for implementing equitable transport policy which is shown to be an important social concern. Transport economists revealed that *anonymous*[§] control policies tend to benefit disproportionately those road users with a high value of time, who are typically rich (Arnott, De Palma and Lindsey, 1994, 1998). Technically, capturing the effects associated with heterogeneity, including collecting required data, introduces a number of difficulties (Newell, 1987; Arnott, De Palma and Lindsey, 1994; Yang and Meng, 1998; Lindsey, 2004) and it remains as a challenging topic in transportation research.

Appendix - Desirable properties of traffic flow models for use in dynamic assignments[**]

Temporal variations of link traffic flows and link travel times in dynamic traffic assignment models are represented by *traffic flow models*. Many different kinds of traffic models have been proposed in the literature (see for example, Vickrey, 1969; Merchant and Nemhauser, 1978a; b; Hendrickson and Kocur, 1981; Mahmassani and Herman, 1984; Newell, 1988; Friesz et al., 1993; Daganzo, 1994, 1995a; Chu, 1995; Ran and Boyce, 1996; Yang and Huang, 1997; Carey

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[‡] Personally, I would recommend this LP approach rather than the state-dependent optimal control approach adopted by Friesz and myself, which involves too much mathematics (e.g. the calculus of variations technique) and too much effort in computation (it is computationally expensive to calculate the time derivatives, i.e. the dynamic externality, and solve the optimal control problem especially we talk about large network).

[§] Anonymous policy refers to the policy which is imposed identically on all individuals.

[**] Although we are going to use the cell transmission model as our underlying traffic flow model, it is still good to know the general requirements on a traffic flow model for use in dynamic traffic assignments.

et al., 2003). Some of these traffic models are more tractable or convenient to use over the others, while some of the models are more realistic representation of traffic dynamics. Because different traffic models produce different estimations for link flows, travel times, and hence solutions of traffic assignments, it is important to understand the properties, plausibility, and applicability of each traffic model. It is also vital to identify the minimum requirements on a traffic model for it to be used in dynamic traffic assignment formulations.

In general, these traffic models can be summarized in the following general form

$$\tilde{c}_a(s) = \rho_a\left[e_a(s), x_a(s), g_a(s)\right], \tag{A-1}$$

where $\tilde{c}_a(s)$ is the link travel time experienced by traffic enters the link *a* at a time *s*. The rate at which traffic enters and leaves the link at time *s* are denoted by $e_a(s)$ and $g_a(s)$ respectively. The amount of traffic present on each link *a* at time *s* is represented by $x_a(s)$. The link travel time is related with the traffic flow quantities through the traffic model $\rho_a(\cdot)$. Daganzo (1995b) showed that for a traffic model which is dependent of inflow, e_a , a sufficiently fast decline in the link inflows can make the traffic model violate first-in-first-out (FIFO) queue discipline. Likewise, Daganzo (1995b) further showed that the traffic model should also be independent of outflow, g_a , because a sufficiently fast decline in the link outflows can also make the traffic model violate FIFO queue discipline in a similar way. Violation of FIFO queue discipline is considered to be unrealistic in a macroscopic travel time model that considers traffic to be flowing continuously, because it implies that the later and faster vehicles will *jump over* the preceding slower vehicles (Carey, 2004a). Following these observations, Daganzo (1995b) suggested that traffic models should only be a function of amount of link traffic, i.e.

$$\tilde{c}_a(s) = \kappa_a\left[x_a(s)\right]. \tag{A-2}$$

Proceeding after Daganzo (1995b), the properties of various kinds of traffic models and their suitability for modelling dynamics of traffic have been investigated widely. Following Carey (2004a; b), and Heydecker and Addison (2005), for plausible estimation of traffic flows and travel times, the link traffic model adopted should possess and satisfy the following five properties:

1. non-negativity;
2. first-in-first-out (FIFO) discipline;
3. conservation of flow;
4. consistency between travel time and flow;
5. causality.

A.1 Non-negativity

The non-negativity principle states that if a positive inflow is loaded into a travel link, then each of the resulting traffic, outflow, and the travel times should always also be positive. This condition can be stated as

$$e_a(s) > 0 \quad \Rightarrow \quad g_a[\tau_a(s)] > 0,\; x_a(s) > 0, [\tau_a(s) - s] > 0, \tag{A-3}$$

where $\tau_a(s)$ is the time of exit for a time of entry at time s, and hence, $\tau_a(s) - s$ is the corresponding travel time along the link.

A.2 First-in-first-out (FIFO) queue discipline

The FIFO queue discipline requires that if a traveller defers his departure time from the origin and join the traffic queue later, then he can expect to arrive at the destination later. That is, the FIFO discipline is satisfied if $s_2 \geq s_1$, $\tau(s_2) \geq \tau(s_1)$ for all times of entry s_1 and s_2. Proposition A.1 then follows for differentiable functions $\tau(\cdot)$.

> ***Proposition A.1:*** If the traffic model satisfies the FIFO queue discipline and the function $\tau(\cdot)$ is differentiable, then the following condition will be satisfied
>
> $$\frac{d\tau}{ds} \geq 0, \tag{A-4}$$
>
> for all times of entry s to the link.
>
> *Proof***:**
> We first have the condition of link FIFO as $\tau(s_2) \geq \tau(s_1)$ for all s_1 and s_2, $s_2 \geq s_1$. This implies that for $\Delta s > 0$, $\dfrac{\tau(s_2) - \tau(s_1)}{s_2 - s_1} = \dfrac{\Delta\tau(s)}{\Delta s} \geq 0$ because both numerator and denominator are positive. Taking the limit on $\Delta s \to 0$ gives $\dfrac{d\tau}{ds} \geq 0$. ☐

The FIFO queue discipline is an essential property for modelling dynamic traffic. Indeed, Daganzo (1995b) and Astarita (1996) have shown that unless the link traffic model respects the FIFO discipline, problems will arise in respect of one or both of non-negativity of traffic and proper propagation of flows. This is further supported by Carey (2004a), who showed that the FIFO discipline is a necessary and sufficient condition to ensure non-negativity of traffic and consistency between traffic flows and corresponding travel times (see proposition 3 in Carey, 2004a). The FIFO condition could be considered to be too strong and unrealistic, but satisfaction of the FIFO discipline is necessary in macroscopic and continuous traffic models. Carey (2004a) explained that FIFO discipline only means to prevent overtaking and passing due to incidental features within the traffic model that do not reflect any real world phenomenon such as a fast vehicle jumps over the preceding slower one.

A.3. Conservation of flow

The conservation of flow states that the traffic volume $x_a(s)$, which is the number of vehicles or the occupancy, on a travel link at any time should be equal to the difference between the cumulative inflow and outflow by that time. The underlying assumption of the principle of conservation is that traffic will neither be generated nor dissipated, for example by vehicles entering from and exiting into side links, within the travel link. However, this assumption could in principle be relaxed by introducing origin or destination nodes to the link as noted by Carey (2004a). This conservation of flow can be written as

$$x_a(s) = E_a(s) - G_a(s), \tag{A-5}$$

where $E_a(s)$ and $G_a(s)$ respectively represent the cumulative inflow and outflow by time *s*. The relationship between the variables in Equation (A-5) is also shown in Figure A.1.

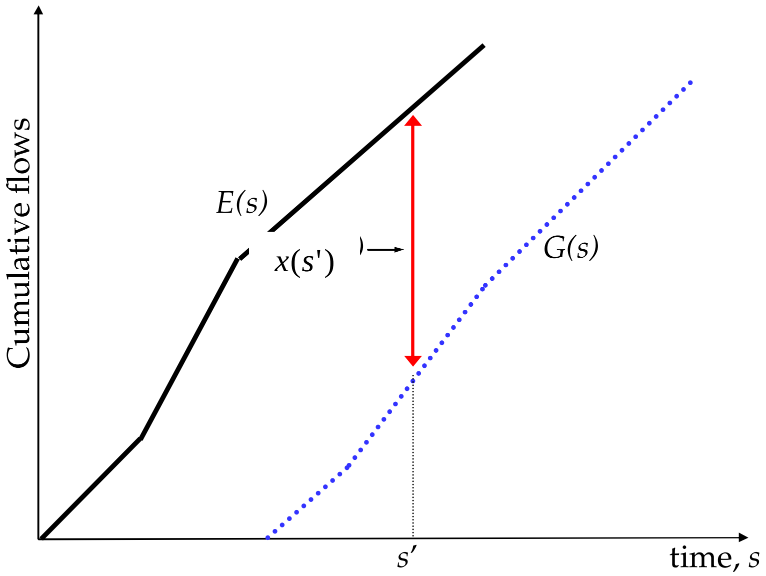

Figure A.1 Flow conservation

If the variables in Equation (A-5) are differentiable with respect to time *s*, then from differentiating (A-5) we have

$$\frac{dx_a(s)}{ds} = e_a(s) - g_a(s). \tag{A-6}$$

Equation (A-6) states that the rate of change of $x_a(s)$ at any time *s* can be determined as the difference between the inflow and the outflow of at that time.

A.4. Consistency between travel time and flow

This travel time-flow consistency is also known as proper propagation of flow (Tobin, 1993; Friesz and Bernstein, 2000; Heydecker and Addison, 2005). It states that the cumulative traffic that has entered up to time s must have exited from the link by exactly time $\tau_a(s)$ (see Figure A.2). This can be expressed as

$$E_a(s) = G_a[\tau_a(s)], \tag{A-7}$$

where $E_a(s)$ and $G_a[\tau_a(s)]$ correspond to the cumulative inflow by s and the cumulative outflow by $\tau_a(s)$ respectively.

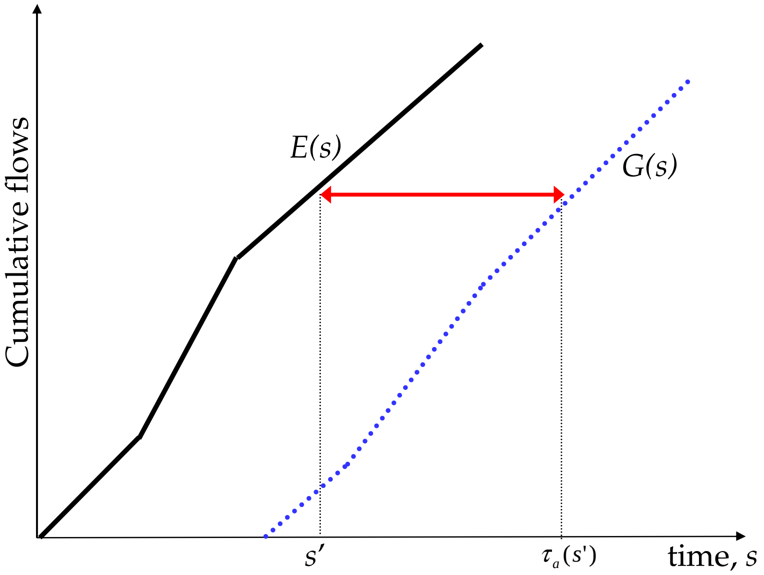

Figure A.2 Consistency of travel-time and traffic-flow

If the variables in Equation (A-7) are differentiable with respect to s, the we can apply the chain rule and differentiate both sides with respect to time s, and hence Equation (A-7) can be written equivalently as

$$e_a(s) = g_a[\tau_a(s)]\frac{d\tau_a(s)}{ds}. \tag{A-8}$$

Equation (A-8) shows the variation of the flow along the travel link should be based on the rate of change of the link travel time, i.e. $\frac{d\tau_a(s)}{ds}$. Following Equation (A-8), a proposition on the non-negativity of link outflow profile is also deduced.

> ***Proposition A.2:*** If the traffic model satisfies FIFO queue discipline, and given a positive profile inflow for all time *s*, then the corresponding profile of outflow is also positive.
>
> *Proof:*

Proposition A.1 shows that FIFO queue discipline implies positive rate of change of link travel time $\frac{d\tau_a(s)}{ds}$ for all time *s*. Proceeding after this and using Equation (A-8), given the link inflow profile $e_a(s)$ and $\frac{d\tau_a(s)}{ds}$ are positive for all time *s*, then the corresponding link outflow profile $g_a[\tau_a(s)]$ must be also positive. □

A.5. Causality

Behaviour of traffic should be affected only by local or conditions downstream, not by traffic conditions upstream. This causal relationship also implies that the outflow profile from a travel link should only depend on the inflow profile at or before the corresponding time of entry but not after.

Cell transmission model

The cell transmission model satisfies all the requirements above and has been applied to dynamic traffic assignment problems (see for example, Lo, 1999; Ziliaskopoulos, 2000; Lo and Szeto, 2002; 2004). Nevertheless, it should be noted that these previous studies revealed that solving the cell transmission model is computationally expensive and may not be feasible for large scale computations. Furthermore, Friesz and Bernstein (2000) also pointed out that the cell transmission model is difficult to analyse mathematically because the outflow function in CTM is piecewise and is not differentiable with respect to its state variable. These are the areas to be remedied in our future research.

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