Dynamic trip assignment

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Abstract

These are notes on trip assignment in TOPL. The aim is to review the standard model, and to start a discussion on TOPL.

Keywords: Dynamic trip assignment; user equilibrium; system optimum; variational inequality.

1 INTRODUCTION

We have a model of a corridor, comprising freeways and arterials. The corridor is modeled as a dynamical system in TOPL. Underlying the dynamical system is a road network described as a graph. A subset of node-pairs is identified as a set of *origin-destination* or OD pairs.

Associated with each OD pair are two entities:

- The *demand profile*—this is a function of time that give for each t the volume (flow) of vehicles that start at the origin at t and want to travel to the destination.
- A *set of routes*—or paths through the network that start at the origin and terminate at the destination.

A *trip assignment* (TA) is an assignment of all OD demand profiles to routes. That is, for each OD pair and time t, the assignment specifies how many of the vehicles will travel over each route associated with the OD pair.

Two TAs are important. A TA is a *user equilibrium* (UE) if no individual vehicle can reduce its travel time given that everyone else follows the TA. A TA is *system optimal* (SO) if it minimizes the total travel time summed over all demand profiles.

Section 2 reviews the standard TA model.

2 STANDARD TA

This discussion is based on (Bergendorff et al. (1997)).

In the standard TA model, demand is stationary, i.e., a constant function of time, so t is omitted. Moreover, as will be seen, there are no dynamics: delay on a link is simply a function of the flow on that link.

$\mathcal{G} = (\mathcal{N}, \mathcal{L})$ is the graph of the road network: $\mathcal{N}$ is the node set, and $\mathcal{L}$ is the set of (directional) links or arcs. A is the node-link incidence matrix:

$$A_{n,l} = \begin{cases} +1, & \text{if link } l \text{ leaves node } n \\ -1, & \text{if link } l \text{ enters node } n \\ 0, & \text{otherwise} \end{cases}, \quad n \in \mathcal{N}, l \in \mathcal{L}.$$

$\mathcal{K} \subset \mathcal{N} \times \mathcal{N}$ is the specified set of OD pairs. Corresponding to each OD pair $k \in \mathcal{K}$ is a demand $b(k)$, regarded as a vector in R^N, $N = |\mathcal{N}|$. The component of $b(k)$ corresponding to the origin node is positive; the component corresponding to the destination is negative but equal in magnitude; all other components of $b(k)$ are zero.

A non-negative flow vector $x(k) \in R^L$ ($L = |\mathcal{L}|$) *supports* OD demands $b(k)$ if $Ax(k) = b(k)$. Suppose $x(k)$ supports demand $b(k)$, $k \in \mathcal{K}$. The corresponding *aggregate* flow vector is $v = \sum_k x(k)$. The set of (aggregate) feasible flows is given by:

$$\begin{aligned} v &= \sum_k x(k) \\ Ax(k) &= b(k), \quad k \in \mathcal{K} \\ x(k) &\geq 0, \quad k \in \mathcal{K}. \end{aligned} \tag{1}$$

Here is an alternative formulation. Let $\mathcal{P}(k)$ be the set of specified paths or routes for OD $k \in \mathcal{K}$. Let $\mathcal{P} = \cup \mathcal{P}(k)$. Let $x(k)$ be the path flow vector indexed by $\mathcal{P}(k)$. Let Γ be the OD pair-path incidence matrix. Then $x(k) \geq 0$ supports $b(k)$ if $\Gamma x(k) = b(k)$. Let Δ be the link-path incidence matrix. Then the set of feasible flows is given by:

$$\begin{aligned} v &= \Delta \sum_k x(k) \\ \Gamma x(k) &= b(k), \quad k \in \mathcal{K} \\ x(k) &\geq 0, \quad k \in \mathcal{K}. \end{aligned} \tag{2}$$

Both (1) and (2) can be converted into the following standard form for feasible flows:

$$\begin{aligned} v &= Zx \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{3}$$

x is called the vector of individual flows and v is the vector of aggregate flows. Define

$$
\begin{aligned}
F &= \{(v, x) \mid v = Zx, Ax = b, x \geq 0\} \\
V &= \{v \mid \exists x : (v, x) \in F\}
\end{aligned}
$$

F is the set of feasible flows, V is the set of feasible aggregate flows.

2.1 Cost function

For each link $l \in \mathcal{L}$, a function $s_l : V \mapsto R_+$ evaluates the *cost* or delay on link l when the aggregate flow (on all links) is v. $s_l(v)$ is the cost incurred on link l by each vehicle on that link when the aggregate flow is v; so the total cost on link l is $s_l(v) \times v_l$, and the *system cost* is

$$
\text{system cost} = \sum_l s_l(v) \times v_l = \langle s(v), v \rangle.
$$

Typically, s_l depends only on v_l, the flow on link l. The most important example is the so-called Bureau of Public Road or *BPR formula*, which for a link of length D and volume f on that link gives the delay

$$
s(f) = \beta D + \alpha T D [1 + \gamma (f/C)^\delta]. \tag{4}
$$

Here T is the free-flow time per unit distance; C is the capacity of the link, f is the flow, f/C is the volume-capacity ratio; $\alpha, \beta, \gamma, \delta$ are parameters ($\delta \approx 4$).[1]

However, it will be useful to allow $s(v)$ to depend on the entire flow vector.

The cost function is (strictly) *monotonic* if

$$
\langle s(v) - s(w), v - w \rangle \geq 0, \quad v, w \in V, \ (> 0, v \neq w).
$$

The cost function for the BPR formula (4) is strictly monotonic when $\delta > 1$.

2.2 UE and SO

$\bar{v} \in V$ is a *user equilibrium* (UE) if it satisfies the *variational inequality*

$$
\langle s(\bar{v}), v - \bar{v} \rangle \geq 0, \quad \forall v \in V. \tag{5}
$$

$\hat{v} \in V$ is *system optimal* (SO) if it minimizes total cost

$$
\langle s(\hat{v}), \hat{v} \rangle \leq \langle s(v), v \rangle, \quad \forall v \in V. \tag{6}
$$

[1] PeMS estimates a BPR curve for any detector.

A UE is said to satisfy Wardrop's first principle according to which each user selects a minimum travel time path or route; a SO satisfies Wardrop's second principle according to which the average travel time is minimized.

If $\bar{v}, \bar{w}$ are UE,

$$\langle s(\bar{v}), \bar{v} - \bar{w} \rangle \leq 0; \quad \langle s(\bar{w}), \bar{w} - \bar{v} \rangle \leq 0;$$

which, upon adding, gives

$$\langle s(\bar{v}) - s(\bar{w}), \bar{v} - \bar{w} \rangle \leq 0,$$

so that if the cost is strictly monotonic, $\bar{v} = \bar{w}$, i.e. the UE is unique, if one exists.

It is not easy to find a UE, i.e., solve the variational inequality (5). *Indeed this will be a major analysis task for TOPL2.* However, it is easy to test whether a proposed $\bar{v}$ is a UE.

Proposition 1 *$\bar{v}$ is a UE if it is a solution of the following linear program (LP):*

$$\begin{aligned} \min_{x,v} \quad & \langle s(\bar{v}), v \rangle \\ v &= Zx \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{7}$$

Hence, by LP duality, $\bar{v}$ is a UE iff there exists ρ such that

$$Z^T s(\bar{v}) \geq A^T \rho \tag{8}$$

$$\langle s(\bar{v}), \bar{v} \rangle = \langle b, \rho \rangle \tag{9}$$

Proposition 2 *$\hat{v}$ is a SO if it is a solution of the following nonlinear program (NP):*

$$\begin{aligned} \min_{x,v} \quad & \langle s(v), v \rangle \\ v &= Zx \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{10}$$

Hence, by KT conditions, if $\hat{v}$ is a SO, there exists ρ such that

$$Z^T [s(\hat{v}) + \nabla s(\hat{v})\hat{v}] \geq A^T \rho \tag{11}$$

$$\langle s(\hat{v}) + \nabla s(\hat{v})\hat{v}, \hat{v} \rangle = \langle b, \rho \rangle. \tag{12}$$

Comparing (8)-(9) and (11)-(12) one sees that a UE $\bar{v}$ cannot be SO unless $\nabla s(\bar{v}) = 0$, i.e., the road network is uncongested.

2.3 Tolls

Suppose a fixed toll β_i is added to the delay $s_l(v)$ over link l, $l \in \mathcal{L}$. Then $\bar{v}_\beta$ is a UE if

$$\langle s(\bar{v}_\beta) + \beta\rangle, v - \bar{v}_\beta\rangle \geq 0, \quad v \in V.$$

By Proposition 1, $\bar{v}_\beta$ is a UE iff there exists ρ such that

$$Z^T[s(\bar{v}_\beta) + \beta] \geq A^T\rho \quad (13)$$

$$\langle s(\bar{v}_\beta) + \beta, \bar{v}_\beta\rangle = \langle b, \rho\rangle \quad (14)$$

Comparing (11)-(12) and (13)-(14) leads to the next result.

Proposition 3 *The SO $\hat{v}$ can be sustained as a UE $\bar{v}_\beta$ with the toll vector*

$$\beta = \nabla s(\hat{v})\hat{v} = \left[\frac{\partial\langle s(v)v\rangle}{\partial v} - s(v)\right]_{v=\hat{v}} = \mathit{MSC}\text{ - }\mathit{AV}. \quad (15)$$

Since β is a money toll, it is assumed in Proposition 3 that the link cost $s_i(v)$ is in money terms as well. This is standardly done by multiplying travel time by *value of time* per unit time, typically \$20-\$30 per hour.

3 TOPL TA

This section discusses various versions of the TA problem for the TOPL or CTM model. As will become clear changes in the CTM model are needed to formulate the TA problem.

Daganzo (1995) gives a CTM model for a network in which each "junction" is one of three types: an *ordinary* junction with one incoming and one outgoing link; a *merge* junction with two incoming and one outgoing link, as in an on-ramp; and a *diverge* junction with two incoming and one outgoing link, as in an off-ramp. Another restriction is that one end of a link must be an ordinary junction, i.e., a link cannot connect a merge and a diverge junction. Two additional specifications are needed:

1. At a merge junction if the downstream constraint (due to capacity or congestion) is smaller than the two upstream flows, one must specify the fraction of each upstream flow that is accommodated. See (Daganzo, 1995, Fig.4).[2]
2. At a diverge junction it is assumed that if either of the outgoing links is constrained, flow from the upstream link is constrained. See (Daganzo, 1995, p. 88).

If we are given OD demands, we cannot take turning movements to be given by split coefficients. We must keep track of the route taken by each vehicle. Thus in a link l the state at time t will be a vector $n_l(t)$ with components $n_l^p(t)$, $p \in \mathcal{P}$, in which $\mathcal{P}$ is the set of routes or paths. Daganzo (1995) proposes in addition to distinguish vehicles by how long they have waited in each link, for the purpose of enforcing FIFO, i.e., in a diverge junction, vehicles that arrived earlier leave sooner. However, we can dispense with this subtlety, and ignore waiting time in a link.

[2](Gomes et al. (2007)), assumes that on-ramp flows have priority.

3.1 Problem 1

Consider our standard freeway, with multiple origins and destinations, Figure 1.

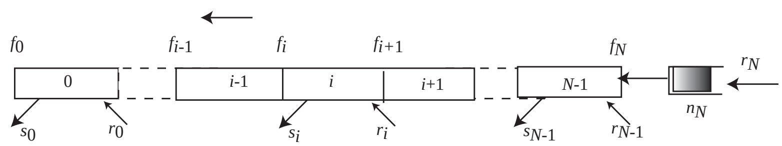

Figure 1: The freeway has N sections. Each section has one on- and one off-ramp.

For each link i, define the N-dimensional vectors

$$\begin{aligned} r_i(k) &= (r_i^0(k), \cdots, r_i^{N-1}(k)), \text{ inflow in } i \text{ destined for } d, \\ s_i(k) &= (s_i^0(k), \cdots, s^{N-1}(k)) \text{ outflow from } i \text{ destined for } d, \\ n_i(k) &= (n_i^0(k), \cdots, n_i^{N-1}(k)), \text{ number of vehicles in } i \text{ destined for } d, \\ f_i(k) &= (f_i^0(k), \cdots, f_i^{N-1}(k)), \text{ flow of vehicles from } i \text{ to } i-1 \text{ destined for } d. \end{aligned}$$

We assume that

$$r_i^d(k) = 0, \ s_i^d(k) = 0, \ d > i,$$

which means that routes don't go backwards; and the initial condition satisfies

$$n_i^d(0) = 0, \ d > i.$$

It should then turn out that

$$n_i^d(k) = 0, \ d > i, \ f_i^d(k) = 0 \ d \geq i.$$

The state at time k is the $N \times N$ array

$$n(k) = \{n_i^d(k), \ 0 \leq i, d \leq N-1\}, \quad k \geq 0.$$

Flow conservation is expressed by

$$n_i(k+1) = n_i(k) - f_i(k) + f_{i+1}(k) + r_i(k) - s_i(k), \ k \geq 0. \tag{16}$$

4 CTM UE

This section formalizes the notion of user equilibrium for a CTM network.

Demand For each OD pair $k \in \mathcal{K}$, $d^k(m), m \geq 0$, is the number of vehicles at time m that appear at the on-ramp $on(k)$ wishing to go to the off-ramp $off(k)$. ($on(k)$ and $off(k)$ are network nodes.)

Trip assignment Associated with $k \in \mathcal{K}$ is a set of *routes* $\mathcal{P}(k)$ from $on(k)$ to $off(k)$. A *trip assignment* for $k \in \mathcal{K}$ is a vector-valued time sequence $\pi^k(m), m \geq 0$, with components $\pi^k_p(m), p \in \mathcal{P}(k)$, such that

$$\pi^k_p(m) \geq 0, \sum_{p \in \mathcal{P}(k)} \pi^k_p(m) = 1. \tag{17}$$

The interpretation is that of the total demand $d^k(m)$ for OD k, $\pi^k_p(m) \times d^k(m)$ vehicles will follow route $p \in \mathcal{P}(k)$, i.e., $\pi^k_p(m)$ is the fraction of trips $d^k(m)$ that are assigned to route p.

Let

$$\Sigma^k(m) = \{\pi^k(m) = \{\pi^k_p(m), p \in \mathcal{P}(k)\} \mid (17) \text{ holds}\};\ \Sigma^k = \prod_{m \geq 0} \Sigma^k(m);\ \Sigma = \prod_{k \in \mathcal{K}} \Sigma^k. \tag{18}$$

Σ is the set of all possible trip assignments. Σ is a convex subset of a finite-dimensional vector space with coordinates indexed by $p \in \mathcal{P}(k)$, $k \in \mathcal{K}$, $m \geq 0$.

Travel time Given demand $D = \{d^k(m), k \in \mathcal{K}\}$ and trip assignment $\pi = \{\pi^k(m), k \in \mathcal{K}\}$, we obtain the route-specific demands at each time m

$$d^k_p(m) = \pi^k_p(m) \times d^k(m),\ k \in \mathcal{K}, p \in \mathcal{P}, m \geq 0. \tag{19}$$

We solve the CTM network model for the demands (19). The solution gives a vector-valued density trajectory denoted $n^{D,\pi}(s), s \geq 0$, with components $n^{D,\pi}_l(s), l \in \mathcal{L}$—the number of vehicles (or density) at time s on link $l \in \mathcal{L}$.

The density is a linear function of the state whose components are $n^{D,\pi}_{l,k,p,m}(s)$, the number of vehicles at time s on link l that come from demand $\pi^k_p(m)d^k(m)$,

$$n^{D,\pi}(s) = \sum_{k,p,m} n^{D,\pi}_{l,k,p,m}(s).$$

Note that the density $n^{D,\pi}_l(s)$ at time s on link l will, in principle, depend on demands at times before and after s for all OD pairs k.

From the fundamental diagram for link l we can obtain the speed $v^{D,\pi}_l(s)$ in link l at time s. Note that this speed depends only on $n^{D,\pi}_l(s)$, i.e.

$$v^{D,\pi}_l(s) = \sigma_l(n^{D,\pi}_l(s)), \tag{20}$$

in which $\sigma_l(n)$ is the speed in link l when its density is n.

Now consider an infinitesimal probe vehicle starting at time m from $on(k)$ and going along route $p \in \mathcal{P}(k)$. We suppose that this vehicle is a particle with no mass so its presence does *not* affect the density and hence the speed on any link.

How much time will it take this particle to reach its destination as it travels along route p?

To answer this question, it will be convenient to make space and time continuous, even though in the CTM model they are both discrete. Suppose route p comprises l_p links, which we arrange in order $0, \cdots, l_p - 1$ beginning with link 0 starting at $on(k)$ and ending with link $l_p - 1$ at $off(k)$. By our convention (Gomes et al. (2007)) each link is 1 mile long. So route p is l_p miles long and we denote by the continuous variable x the distance along this route. We will say that x is on link l if $x \in [l, l+1)$. Similarly, the discrete time s will be associated with the continuous time interval $t \in [s\delta, (s+1)\delta)$, in which δ is the period.

At any continuous distance x and time t *define* the speed to be

$$v(x,t) = v_l^{D,\pi}(s), \;\; l \leq x \leq l+1, t \in [s\delta, (s+1)\delta), \tag{21}$$

i.e., we define the speed on link l at time $t \in [s\delta, (s+1)\delta)$ as $v_l^{D,\pi}(s)$.

We suppose that our probe particle moves with speed given by (21), so its position $x(t)$ at time t follows the differential equation

$$\frac{dx}{dt}(t) = v(x(t), t), \;\; t \geq m\delta, \;\; x(m\delta) = 0. \tag{22}$$

We can then define the travel time of the particle as

$$T_p^{D,\pi}(m) = \inf\{t \mid x(t) = l_p\} - m\delta, \tag{23}$$

which is the first time that the particle reaches the end of path p after traveling l_p miles, which is the length of p.

We define $T_p^{D,\pi}(m)$ as the travel time of a 'user' who starts at $on(k)$ at time m and takes route p when the overall demand and trip assignment is (D, π).

We call $v(x,t), 0 \leq x \leq l_p - 1; m\delta \leq t < \infty$ the *velocity field*. We can of course define its reciprocal, the *time delay field*

$$\Delta(t,x) = [v_l^{D,\pi}(s)]^{-1}, \;\; l \leq x \leq l+1, t \in [s\delta, (s+1)\delta), \tag{24}$$

and consider the solution $\tau(x)$ of the differential equation

$$\frac{d\tau}{dx}(x) = \Delta(\tau, x), \;\; x \geq 0, \;\; \tau(0) = m\delta. \tag{25}$$

We need to check that the travel time (23) is also given by

$$T_p^{D,\pi}(m) = \tau(l_p) - m\delta. \tag{26}$$

Technical issues The velocity and delay fields are discontinuous, piecewise constant functions of time and space. So they are certainly not Lipschitz functions. Hence we need a separate argument to prove that the solutions of (22), (25) exist, are unique, and vary continuously with initial conditions. To see what this involves, Figure 2 depicts the field $v(x,t)$ and three trajectories.

It seems clear from the figure that the solutions exist, are unique, have piecewise constant slope and vary continuously with initial conditions. The only hitch can occur if the speed is

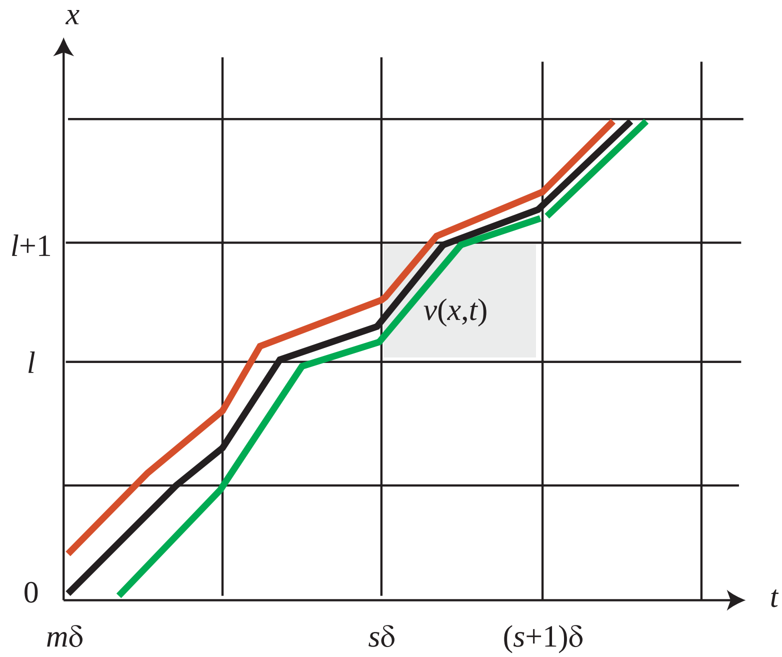

Figure 2: The field $v(x,t)$ is piecewise constant.

zero. In that case a trajectory may lie along a discontinuity boundary, and we certainly will not have continuity with initial conditions and uniqueness is also problematic. However, if the speed is bounded away from 0, all trajectories will go 'through' the discontinuity boundary in zero time, i.e., the trajectories will be 'transversal' to the discontinuity. This suggests the following proposition, which needs proof.

Proposition 4 *Suppose the CTM speeds* $v_l^{D,\pi}(s) \geq v_0 > 0$, $l = 0, \cdots, l_p - 1$, $s \geq 0$. *Then solutions to (22), (25) exist, are unique, and vary continuously with initial conditions. Moreover, (9) holds.*

User Equilibrium Consider a user who forms part of the demand $d^k(m)$. If this user knows the travel times along all possible routes $T_p^{D,\pi}(m), p \in \mathcal{P}(k)$, he will choose a route that minimizes his travel time, i.e., a route in

$$\mathcal{P}^{D,\pi}(m) = \arg\min\{T_p^{D,\pi}(m) \mid p \in \mathcal{P}(k)\}. \tag{27}$$

If all users constituting the demand $d^k(m)$ do this, they will end up with a trip assignment $\hat{\pi}^k(m)$ such that

$$\hat{\pi}_p^k(m) = 0, \; p \notin \mathcal{P}^{D,\pi}(m).$$

That is, these users will follow a trip assignment from

$$\Pi^k(m) = \{\hat{\pi}^k(m) \mid \hat{\pi}_p^k(m) = 0, \; p \notin \mathcal{P}^{D,\pi}(m)\}. \tag{28}$$

Since the constraints in (28) are linear, $\Pi^k(m)$ is a closed, convex subset of $\Sigma^k(m)$.

Fix demand $D = \{d^k(m)\}$. We say that a trip assignment $\pi = \{\pi^k(m)\} \in \Sigma$ is a *user equilibrium* if

$$\pi^k(m) \in \Pi^k(m), \quad k \in \mathcal{K}, m \geq 0. \tag{29}$$

Fixed point Define the point-to-convex set map or correspondence $\Theta : \Sigma \mapsto 2^\Sigma$ by

$$\Theta(\{\pi^k(m)\}) = \{\{\hat{\pi}^k(m)\} \mid \hat{\pi}^k(m) \in \Pi^k(m)\}. \tag{30}$$

Then, a trip assignment $\pi \in \Sigma$ is **a user equilibrium** if and only if it is a fixed point of Θ, i.e.,

$$\pi \in \Theta(\pi). \tag{31}$$

The correspondence Θ is *upper hemicontinuous* if for convergent sequences $\pi_n \to \pi$ and $\hat{\pi}_n \to \hat{\pi}$ in Σ,

$$\forall n, \hat{\pi}_n \in \Theta(\pi_n) \Rightarrow \hat{\pi} \in \Theta(\pi).$$

Kakutani fixed point theorem Suppose Σ is compact convex and $\Theta : \Sigma \mapsto 2^\Sigma$ is a closed convex upper hemicontinuous correspondence. Then Θ has a fixed point.

Thus to prove the existence of a user equilibrium (29) it is enough to prove that the correspondence (30) is upper hemicontinuous, since it is already compact and convex. We proceed to show this.

Fix demand D. The state of the system at discrete time s is the vector $n(s)$ with components $n_{l,k,p}(s), l \in \mathcal{L}, k \in \mathcal{K}, p \in \mathcal{P}$. This state evolves according to the difference equation

$$n^\pi_{l,k,p}(s+1) = f^\pi_{l,k,p}(n^\pi(s)), \quad 0 \leq s \leq T, \tag{32}$$

with a specified initial condition $n(0)$. In (32), $\pi = \{\pi^k\}$ is the trip assignment. We treat π as a 'parameter' of the difference equation.

Proposition 5 *The right-hand side $f^\pi_{l,k,p}(n(s))$ of (32) is a* continuous *function of its arguments and the parameter π.*

Proposition 5 is true for the CTM model of a single freeway and a quick look at Daganzo (1995) suggests it will hold. In any case this, too, needs proof.

Define the 'output' of (32) as the aggregate density vector with components (see (20))

$$n^\pi_l(s) = \sum_{k,p} n^\pi_{l,k,p}(s), \quad l \in \mathcal{L},$$

and the corresponding aggregate speed vector with components

$$v^\pi_l(s) = \sigma_l(n^\pi_l(s)) \;\; l \in \mathcal{L},$$

Proposition 6 *Suppose the trip assignment* π *is such that* $v_l^\pi(s) \geq v_0 > 0$ *for all* l, s. *Then for all paths* p *and starting times* m *the travel time* $T_p^\pi(m)$ *is a continuous function at* π.

Proof This should follow from Proposition 4. Let $\Delta^\pi(t,x)$ be the time delay field for a particle traveling along path p, starting at time m when π is the trip assignment. By choosing $|\hat{\pi} - \pi|$ small enough we can guarantee that $|\Delta^{\hat{\pi}}(t,x) - \Delta^\pi(t,x)|$ is small. Now show that the solutions $\tau^{\hat{\pi}}(x)$ and $\tau^\pi(x)$ of (25) are close enough. This last statement needs proof. □

Proposition 7 *The map* Θ *is upper hemicontinuous.*

Proof Suppose $\pi_n \to \pi$, $\hat{\pi}_n \to \hat{\pi}$ and $\hat{\pi}_n \in \Theta(\pi_n)$ for all n. We must show that (see (28))

$$\hat{\pi}_p^k(m) > 0 \Rightarrow T_p^\pi(m) \leq T_q^\pi(m), \forall q \in \mathcal{P}(k). \tag{33}$$

Since $\hat{\pi}_p^k(m) > 0$ it must be the case that $\hat{\pi_n}_p^k(m) > 0$ for all large n, and since $\hat{\pi_n} \in \Theta(\pi_n)$, so for all n

$$T_p^{\pi_n}(m) \leq T_q^{\pi_n}(m), \ \forall q \in \mathcal{P}(k),$$

which implies (33) by continuity of T (Proposition 6). □

Theorem 4.1 *There exists a UE for the CTM model.*

Proof By the Kakutani fixed point theorem there is a fixed point (31). □

5 QUEUES

Recall from Section 2.1 that the cost (delay) $s(v)$ on a link with flow v is monotonic if $(s(v) - s(w))(v - w) \geq 0$. If $v, s(v)$ are real-valued, this is equivalent to saying that $v \mapsto s(v)$ is a monotonically increasing function.

In this section I intend to show that for a *point queue* model the delay is monotonic.

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