Calculation of HOV flows in Aurora

Alex Kurzhanskiy

Draft for the 10/09/2008 TOPL meeting

Problem setting

Consider a freeway node with m input and n output links. Each link accounts for HOV and SOV traffic.

Notation:

ρ_k^{HOV} HOV portion of the density in link k;

ρ_k^{SOV} SOV portion of the density in link k;

$\rho_k = \rho_k^{HOV} + \rho_k^{SOV}$ total density in link k;

v_k free flow speed in link k;

w_k congestion wave speed in link k;

$\overline{\rho}_k$ jam density in link k;

F_k capacity of link k;

h_{ij} HOV flow going from the i-th input link to the j-th output link;

s_{ij} SOV flow going from the i-th input link to the j-th output link;

$\mathcal{A} = \{\alpha_{ij}\}$ split ratio matrix for HOV flows;

$\mathcal{B} = \{\beta_{ij}\}$ split ratio matrix for SOV flows.

In case of HOV and ordinary freeway inputs and outputs, matrices $\mathcal{A}$ and $\mathcal{B}$ are generally not known. We would like to find h_{ij} and s_{ij} values that maximize the total flow leaving the node, namely

$$\sum_{i,j}(h_{ij} + s_{ij}) \rightarrow \max .$$

This linear program must be solved under the following constraints:

1. upstream demand:

$$\sum_{j=1}^{n} h_{i,j} \leq v_i \rho_i^{HOV} \quad \text{and} \quad \sum_{j=1}^{n} s_{i,j} \leq v_i \rho_i^{SOV}, \quad i = 1..m;$$

2. downstream capacity:

$$\sum_{i=1}^{m} (h_{ij} + s_{ij}) \leq F_j \quad \text{and} \quad \sum_{i=1}^{m} (h_{ij} + s_{ij}) \leq w_j(\overline{\rho}_j - \rho_j), \quad j = 1..n;$$

3. flows cannot be negative:

$$h_{ij} \geq 0 \quad \text{and} \quad s_{ij} \geq 0, \quad i = 1..m, \quad j = 1..n;$$

4. FIFO condition[1]:

$$\rho_i^{SOV} \sum_{j=1}^{n} h_{ij} = \rho_i^{HOV} \sum_{j=1}^{n} s_{ij}, \quad i = 1..m.$$

The original linear program may have multiple solutions, and additional constraints reduce the set of solutions. Additional constraints are introduced if certain elements of matrices $\mathcal{A}$ and $\mathcal{B}$ are known. For example, if we know that input $\hat{i}$ and output $\hat{j}$ are HOV links, then additional constraints will be

$$s_{\hat{i}j} = 0, \quad j = 1..n, \quad \text{and} \quad s_{i\hat{j}} = 0, \quad i = 1..m.$$

Once we pick an LP solution (h_{ij}, s_{ij}), $i = 1..m$ and $j = 1..n$, we define split ratios as

$$\alpha_{ij} = \frac{h_{ij}}{\sum_{j=1}^{n} h_{ij}} \quad \text{and} \quad \beta_{ij} = \frac{s_{ij}}{\sum_{j=1}^{n} s_{ij}}.$$

Implementation of the LP procedure in Aurora can be done using *OR-Objects* Java library (*http://opsresearch.com/OR-Objects*).

When split ratio matrices $\mathcal{A}$ and $\mathcal{B}$ are known, we find h_{ij} and s_{ij}, $i = 1..m$ and $j = 1..n$, are computed the way it is currently implemented in Aurora. To account for separate HOV and SOV flows, we treat the node as if it has $2m$ (m HOV and m SOV) input links and the split ratio matrix has the form $\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}$. Additionally, constraint 4 must be enforced.

---

[1] This condition plays the role of split ratios when they are known, preventing a link to generate HOV flow when SOV flow is blocked.

Example

A freeway node with HOV, ordinary freeway and on-ramp inputs and HOV, ordinary freeway and off-ramp outputs is shown in Figure 1. No SOVs are allowed in the HOV links. Split ratios for the off-ramp are known: they are denoted $\hat{\alpha}_{i3}$ and $\hat{\beta}_{i3}$, $i = 1..3$.

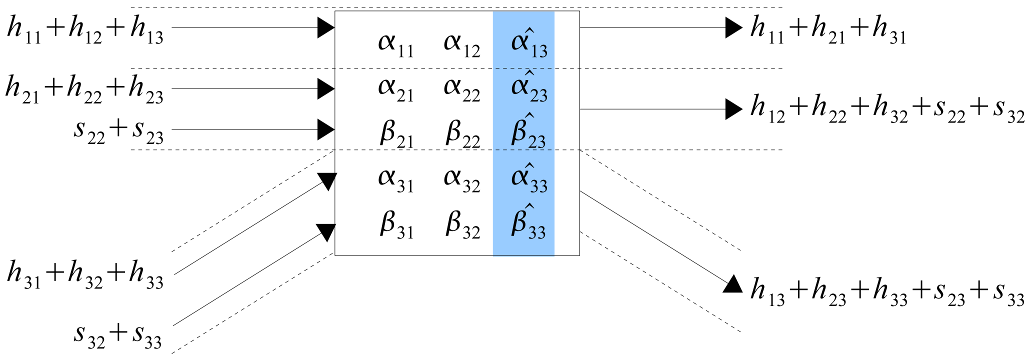

Figure 1: Freeway node with HOV, ordinary freeway and on-ramp input links, and HOV, ordinary freeway and off-ramp output links.

The LP is set as described above. The additional constraints are

5. no SOVs are allowed on the incoming HOV link:

$$s_{1j} = 0, \quad j = 1..3;$$

6. no SOVs are allowed on the outgoing HOV link:

$$s_{i1} = 0, \quad i = 1..3;$$

7. off-ramp split ratios are known:

$$h_{i3} = \frac{\hat{\alpha}_{i3}}{1 - \hat{\alpha}_{i3}}(h_{i1} + h_{i2}) \quad \text{and} \quad s_{i3} = \frac{\hat{\beta}_{i3}}{1 - \hat{\beta}_{i3}} s_{i2}, \quad i = 1..3.$$

Drawback of LP

The problem with LP is that in case of multiple solutions, the choice of solution depends on the particular solver or its settings.

For example, consider the following scenario for the two-by-two node shown in Figure 2, where SOVs are allowed only on links 2 and 4.

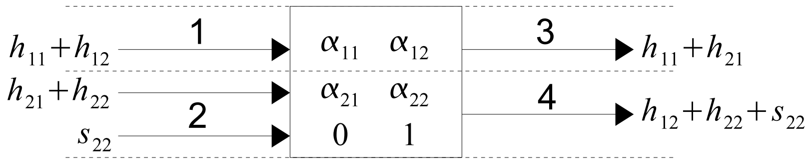

Figure 2: Freeway node with two inputs and two outputs: links 1 and 3 are HOV, links 2 and 4 are ordinary freeway links.

Fundamental diagrams for the HOV and ordinary links are presented in Figure 3. Links 1, 2 and 4 are in free flow. Ordinary links have low densities. Density in link 3 is slightly above critical.

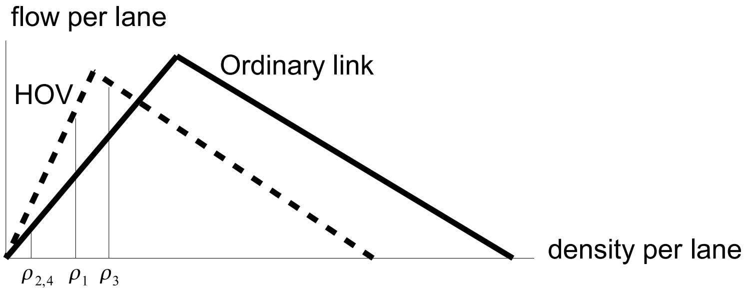

Figure 3: Fundamental diagrams for HOV and ordinary freeway links: HOV link has greater free flow speed and smaller capacity.

In this situation LP has multiple solutions. An LP solver, however, picks just one, which results in

$$\alpha_{11} = 1 \qquad \alpha_{12} = 0$$
$$\alpha_{21} = 1 \qquad \alpha_{22} = 0.$$

The question is: is there a *procedure* that computes input-output flows whose result is also one of possibly many LP solutions?

Procedure for input/output flow computation

Denote $\gamma_i = \rho_i^{HOV}/\rho_i^{SOV}$ (here we assume both $\rho_i^{HOV}, \rho_i^{SOV} \neq 0$), $i = 1..m$.

Known $\mathcal{A}$ and $\mathcal{B}$

1. Compute $\hat{h}_{ij}$ and $\hat{s}_{ij}$ for the split ratio matrix $\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}$ following the Aurora procedure.
2. Impose constraint 4 from the original LP:
   if $\sum_{j=1}^{n} \hat{h}_{ij} < \gamma_i \sum_{j=1}^{n} \hat{s}_{ij}$, then
   $$h_{ij} = \hat{h}_{ij} \quad \text{and} \quad s_{ij} = h_{ij}/\gamma_i,$$
   otherwise
   $$s_{ij} = \hat{s}_{ij} \quad \text{and} \quad h_{ij} = \gamma_i s_{ij},$$
   for $i = 1..m$ and $j = 1..n$.

Unknown $\mathcal{A}$ and $\mathcal{B}$

1. Compute total upstream demand
   $$D = \sum_{i=1}^{m} \rho_i v_i$$
2. Compute total output capacity
   $$C = \sum_{j=1}^{n} \min\{F_j, w(\overline{\rho}_j - \rho_j)\}.$$
3. Find a solution of linear equation
   $$\begin{aligned} \sum_{j=1}^{n} s_{ij} &= \frac{\rho_i v_i}{1+\gamma_i} \min\left\{1, \frac{C}{D}\right\}, \quad i = 1..m, \\ \gamma_i s_{ij} - h_{ij} &= 0, \quad i = 1..m, \ j = 1..n. \end{aligned}$$
   In case there are infinitely many solutions, we generate one by first filling up the capacity of output link 1, then 2, and so on.

Partially known $\mathcal{A}$ and $\mathcal{B}$

Suppose, for $j = 1..n_1$ all the split ratios are known exactly, $\alpha_{ij} = \hat{\alpha}_{ij}$ and $\beta_{ij} = \hat{\beta}_{ij}$, $i = 1..m$; for $j = n_1 + 1..n_2$ some of the split ratios are known; and for $j = n_2 + 1..n$ split ratios are not known.

Define demand from the input i to the output j:

$$d_{ij} = d_{ij}^{HOV} + d_{ij}^{SOV},$$

where

$$d_{ij}^{HOV} = \alpha_{ij}\rho_i^{HOV} \quad \text{and} \quad d_{ij}^{SOV} = \beta_{ij}\rho_i^{SOV}.$$

1. Set $d_{ij}^{(1)} = d_{ij}$ for $i = 1..m$ and $j = 1..n$.
2. Set $D^{(1)} = D$, where $D = \sum_{i=1}^{m} \rho_i v_i$.
3. For $j = 1..n_1$
    (a) compute demand for link j:
    $$d_j^{(j)} = \sum_{i=1}^{m} d_{ij}^{(j)};$$
    (b) compute capacity of link j:
    $$c_j = \min\{F_j, w(\overline{\rho}_j - \rho_j)\};$$
    (c) adjust the demand:
    $$d_{ik}^{(j+1)HOV} = \min\left\{1, \frac{c_j}{d_j^{(j)}}\right\} d_{ik}^{(j)HOV} \quad \text{and} \quad d_{ik}^{(j+1)SOV} = \min\left\{1, \frac{c_j}{d_j^{(j)}}\right\} d_{ik}^{(j)SOV}$$
    for those i and j, for which the split ratios are known, $1 \leq i \leq m$ and $1 \leq j \leq n$.

    (d) adjust total input demand:
    $$D^{(j+1)} = \min\left\{1, \frac{c_j}{d_j^{(j)}}\right\} D^{(j)}.$$
4. For $j = n_1 + 1..n_2$
    (a) compute partial adjusted demand $d_j^{(j)}$ for link j counting only those entries for which split ratios are known, and capacity c_j;

    (b) if $c_j \leq d_j^{(j)}$, set all the unknown split ratios to 0 and adjust the demand as in step 2c and 2d, otherwise set some or all of the unknown split ratios so as to maximize the demand without exceeding c_j and call the new adjusted demand $d_{ij}^{(j+1)}$, $i = 1..m$, and $D^{(j+1)} = D^{(j)}$.
5. Compute the total remaining adjusted input demand
$$D^{-} = D^{(n_2+1)} - \sum_{i=1}^{m} \sum_{j=1}^{n_2} d_{ij}^{(n_2+1)}.$$
6. Compute the remaining capacity
$$C^{-} = \sum_{j=n_2+1}^{n} \min\{F_j, w(\overline{\rho}_j - \rho_j)\}.$$
7. Set
$$h_{ij} = \min\left\{1, \frac{C^{-}}{D^{-}}\right\} d_{ij}^{(n_2+1)HOV} \quad \text{and} \quad s_{ij} = \min\left\{1, \frac{C^{-}}{D^{-}}\right\} d_{ij}^{(n_2+1)SOV}, \quad i = 1..m, \ j = 1..n_2.$$
8. For $j = n_2 + 1..n$, the remaining adjusted input demand, the remaining capacity, and bounds on split ratios $\alpha_{ij} \in [0, (1 - \sum_{k=1}^{n_2} \hat{\alpha}_{ik})]$ and $\beta_{ij} \in \left[0, \left(1 - \sum_{k=1}^{n_2} \hat{\beta}_{ik}\right)\right]$, compute h_{ij} and s_{ij} as in the subsection about unknown $\mathcal{A}$ and $\mathcal{B}$.